∞-Categorical Generalized Langlands Program I: Mixed-Parity Modules and Sheaves

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Abstract

Mixed-parity module emerges for instance when a de Rham Galois representation is being tensored with a square root of cyclotomic character, which produces half odd integers as the corresponding Hodge-Tate weights. We build the whole foundation on the p-adic Hodge theory in this setting over small v-stacks after Scholze and we also consider certain moduli v-stack which parametrizes families of mixed-parity Hodge modules. Examples of the small v-stacks in our mind are rigid analytic spaces over p-adic fields and moduli v-stack of vector bundles over Fargues-Fontaine curves. The preparation implemented at this level will be expected to provide further essential foundationalization for generalized Langlands program after Langlands, Drinfeld, Fargues-Scholze. One side of the generalized Langlands correspondence in the geometric setting is the perverse motivic derived ∞-category over Moduli_G related to smooth representations of reductive groups, while the other side of the generalized Langlands correspondence in the geometric setting is the corresponding derived ∞-category over the stack of mixed-parity L-parametrizations (i.e. from two-fold covering of the Weil group into ℓ-adic groups) related to the representations of Weil group in our setting into Langlands dual groups. Although after Scholze and Fargues-Scholze our generalized Langlands program can go along ℓ-adic cohomologicalization to immediately achieve various solid derived ∞-categories $\text{DerCat}_{\text{ét}}(\text{Moduli}_G, \square)$, $\text{DerCat}_{\text{lisse},\blacksquare}(\text{Moduli}_G, \square)$, $\text{DerCat}_{\blacksquare}(\text{Moduli}_G, \square)$ and so on with well-established formalism regarding 6-functors, we already provide certain p-adic cohomologicalization of the story over Moduli_G.

Contents

Reference 1.

- □ Chapter 1 Main References: [Sch1], [KL1], [KL2], [BL1], [BL2], [BS], [BHS], [Fon1], [CS1], [CS2], [BK], [BBK], [BBBK], [KKM], [BBM], [LZ], [TT], [B], [Shi], [AI1], [AI2], [AI3], [AB1], [AB2], [Fon2], [Fon3], [Fa1], [M], [Fa2], [Fa3], [Fon4], [Fon5], [Fon6];

- □ Chapter 2 Main References: [Sch1], [Sch2], [FS], [KL1], [KL2], [BL1], [BL2], [BS], [BHS], [Fon1], [CS1], [CS2], [BK], [BBK], [BBBK], [KKM], [BBM], [LZ], [TT], [B], [Shi], [AI1], [AI2], [AI3], [AB1], [AB2], [Fon2], [Fon3], [Fa1], [M], [Fa2], [Fa3], [Fon4], [Fon5], [Fon6];

- □ Chapter 3 Main References: [Sch1], [Sch2], [FS], [FF], [KL1], [KL2], [BL1], [BL2], [BS], [BHS], [Fon1], [CS1], [CS2], [BK], [BBK], [BBBK], [KKM], [BBM], [LBV], [B], [SW]; [Sch1], [Sch2], [FS], [KL1], [KL2], [BL1], [BL2], [BS], [BHS], [Fon1], [CS1], [CS2], [BK], [BBK], [BBBK], [KKM], [BBM], [LZ], [TT], [B], [Shi], [AI1], [AI2], [AI3], [AB1], [AB2], [Fon2], [Fon3], [Fa1], [M], [Fa2], [Fa3], [Fon4], [Fon5], [Fon6];

- □ Chapter 4 Main References I: [FS], [FF], [Sch1], [Sch2], [KL1], [KL2], [LBV], [B], [SW], [BS], [Lan1], [Drin1], [Drin2], [Zhu], [DHKM];

- □ Chapter 4 Main References II: [pHodgeT], [pHodgeF], [pHodgeS1], [pHodgeS2], [pHodgeKL1], [pHodgeKL2], [pHodgeBS], [pHodgeKPX], [pToAnCS1], [pToAnCS2], [pToAnCS3], [pToAnCS4], [pToAnBBBK], [LPL], [LPD1], [LPLL], [LPVL], [LPC], [LPFS], [LPGL], [LPEGH], [LPEG], [LPZ], [LPDHKM], [LPD2], [LPL], [LPD1], [LPLL], [LPVL], [LPC], [LPFS], [LPGL], [LPEGH], [LPEG], [LPZ], [LPDHKM], [LPD2].

Notations:

- Chapter 1: The period sheaves in the pro-étale topology in this chapter are assumed to be already tensored with a finite extension of $\mathbb{Q}_p$ containing square roots of p, although we do write that notation in an explicit way. We assume the corresonding interval I contains 1.

$$\Gamma_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}, \Gamma^{O}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}, \tag{1}$$

$$\Gamma_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}, \Gamma^{O}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}, \tag{2}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}, \tag{3}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2},\log(t)\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2},\log(t)\}; \tag{4}$$

$$\Gamma_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}, \Gamma^{O}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}, \tag{5}$$

$$\Gamma_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}, \Gamma^{O}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}, \tag{6}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}, \tag{7}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2},\log(t)\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2},\log(t)\}. \tag{8}$$

- Chapter 2: The period sheaves in the v-topology in this chapter are assumed to be already tensored with a finite extension of $\mathbb{Q}_p$ containing square roots of p, although we do write that notation in an explicit way. We assume the corresonding interval I contains 1.

$$\Gamma_{\mathrm{deRham},X,v}\{t^{1/2}\}, \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}, \tag{9}$$

$$\Gamma_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\}, \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\}, \tag{10}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{11}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2},\log(t)\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2},\log(t)\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2},\log(t)\}; \tag{12}$$

$$\Gamma_{\mathrm{cristalline},X,v}\{t^{1/2}\}, \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}, \tag{13}$$

$$\Gamma_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}, \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}, \tag{14}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{15}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2},\log(t)\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2},\log(t)\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2},\log(t)\}. \tag{16}$$

- Chapter 3: The period rings in 3.1, 3.2 are assumed to be not already tensored with a finite extension of $\mathbb{Q}_p$ containing square roots of p, we do write that notation in an explicit way; Then the period sheaves in the v-topology in 3.3, 3.4, 3.5 are assumed to be already tensored with a finite extension of $\mathbb{Q}_p$ containing square roots of p, although we do write that notation in an explicit way. We assume the corresonding interval I contains 1.

$$\Gamma_{\mathrm{deRham},X,v}\{t^{1/2}\}, \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}, \tag{17}$$

$$\Gamma_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\}, \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\}, \tag{18}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{19}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2},\log(t)\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2},\log(t)\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2},\log(t)\}; \tag{20}$$

$$\Gamma_{\text{cristalline},X,v}\{t^{1/2}\}, \Gamma^{O}_{\text{cristalline},X,v}\{t^{1/2}\}, \tag{21}$$

$$\Gamma_{\text{cristalline},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{O}_{\text{cristalline},X,v}\{t^{1/2}, \log(t)\}, \tag{22}$$

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}, \tag{23}$$

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}, \log(t)\}. \tag{24}$$

- Chapter 4: The period rings in this chapter are assumed to be not already tensored with a finite extension of $\mathbb{Q}_p$ containing square roots of p, we do write that notation in an explicit way.

Chapter 1

Mixed-Parity p-adic Hodge Modules over Pro-Étale Sites

1.1 Geometric Family of Mixed-Parity Hodge Modules I: de Rham Situations

1.1.1 Period Rings and Sheaves

Reference 2. [Sch1], [KL1], [KL2], [BL1], [BL2], [BS], [BHS], [Fon1], [CS1], [CS2], [BK], [BBK], [BBBK], [KKM], [BBM], [LZ], [M].

Rings

Let X be a rigid analytic space over $\mathbb{Q}_p$. We have the corresponding étale site and the corresponding pro-étale site of X, which we denote them by $X_{\text{proét}}, X_{\text{ét}}$. The relationship of the two sites can be reflected by the corresponding morphism $f : X_{\text{proét}} \longrightarrow X_{\text{ét}}$. Then we have the corresponding de Rham period rings and sheaves from [Sch1]:

$$\Gamma_{\text{deRham},X,\text{proét}}, \Gamma^{O}_{\text{deRham},X,\text{proét}}. \tag{1.1}$$

Our notations are different from [Sch1], we use Γ to mean B in [Sch1], while Γ^{O} will be the corresponding OB ring in [Sch1].

Definition 1. Now we assume that $p > 2$, following [BS] we join the square root of t element in $\Gamma_{\text{deRham},X,\text{proét}}$ which forms the sheaves:

$$\Gamma_{\text{deRham},X,\text{proét}}\{t^{1/2}\}, \Gamma^{O}_{\text{deRham},X,\text{proét}}\{t^{1/2}\}. \tag{1.2}$$

And following [BL1], [BL2], [Fon1], [BHS] we further have the following sheaves of rings:

$$\Gamma_{\text{deRham},X,\text{proét}}\{t^{1/2}, \log(t)\}, \Gamma^{O}_{\text{deRham},X,\text{proét}}\{t^{1/2}, \log(t)\}. \tag{1.3}$$

Definition 2. We use the notations:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét}}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I} \tag{1.4}$$

to denote the perfect Robba rings from [KL1], [KL2], where $I \subset (0, \infty)$. Then we join $t^{1/2}$ to these sheaves we have:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét}}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}. \tag{1.5}$$

And following [BL1], [BL2], [Fon1], [BHS] we have the following larger sheaves:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét}}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}, \log(t)\}. \tag{1.6}$$

Definition 3. From now on, we use the same notation to denote the period rings involved tensored with a finite extension of $\mathbb{Q}_p$ containing square root of p as in [BS].

$$\Gamma_{\text{deRham},X,\text{proét}}\{t^{1/2}\}, \Gamma^{O}_{\text{deRham},X,\text{proét}}\{t^{1/2}\}. \tag{1.7}$$

$$\Gamma_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\},\Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}. \tag{1.8}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}. \tag{1.9}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2},\log(t)\}. \tag{1.10}$$

This is necessary since we to extend the action of φ to the period rings by $\varphi(t^{1/2}\otimes 1)=\varphi(t)^{1/2}\otimes 1$.

Modules

We consider quasicoherent presheaves in the following two situation:

- □ The solid quasicoherent modules from [CS1], [CS2];
- □ The ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM] with the corresponding monomorphic ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM].

Definition 4. We use the notation:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.11}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Definition 5. We use the notation:

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}}, \tag{1.12}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.13}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.14}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of inductive Banach modules.

Definition 6. We use the notation:

$$\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.15}$$

to denote the $(\infty,1)$-categories of solid quasicoherent sheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Mixed-Parity Hodge Modules without Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Definition 7. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}, \tag{1.16}$$

we consider the following functor dR sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}) \tag{1.17}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}). \tag{1.18}$$

We call F mixed-parity de Rham if we have the following isomorphism[1]:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\} \overset{\sim}{\longrightarrow} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\} \tag{1.19}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\} \overset{\sim}{\longrightarrow} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}. \tag{1.20}$$

Definition 8. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}, \tag{1.21}$$

we consider the following functor $\mathrm{dR}^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}) \tag{1.22}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}). \tag{1.23}$$

We call F mixed-parity almost de Rham if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\} \tag{1.24}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\} \tag{1.25}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\} \tag{1.26}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}. \tag{1.27}$$

[1] As in [KL, Definition 10.10], when we consider the corresponding de Rham, cristalline, semi-stable functors we will assume 1 is belonging to the interval I in all the following corresponding discussion.

We now define the $(\infty,1)$-categories of mixed-parity de Rham modules and he corresponding mixed-parity almost de Rham modules by using the objects involved to generated these categories:

Definition 9. Considering all the mixed parity de Rham bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.28}$$

generated by the mixed-parity de Rham bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity de Rham complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.29}$$

Definition 10. Considering all the mixed parity almost de Rham bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.30}$$

generated by the mixed-parity almost de Rham bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity de Rham complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.31}$$

Then the corresponding mixed-parity de Rham functors can be extended to these categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}, \tag{1.32}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.33}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.34}$$

Mixed-Parity Hodge Modules with Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Remark 1. All the coherent sheaves over mixed-parity Robba sheaves in this section will carry the corresponding Frobenius morphism $\varphi: F \xrightarrow{\sim} \varphi^* F$.

Definition 11. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}, \tag{1.35}$$

we consider the following functor dR sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}) \tag{1.36}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}). \tag{1.37}$$

We call F mixed-parity de Rham if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}) \otimes \Gamma^{O}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\} \overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\} \tag{1.38}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}) \otimes \Gamma^{O}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\} \overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}. \tag{1.39}$$

Definition 12. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}, \tag{1.40}$$

we consider the following functor $\mathrm{dR}^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}, \log(t)\}) \tag{1.41}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}, \log(t)\}). \tag{1.42}$$

We call F mixed-parity almost de Rham if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{O}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}, \log(t)\} \tag{1.43}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}, \log(t)\} \tag{1.44}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{O}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}, \log(t)\} \tag{1.45}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}, \log(t)\}. \tag{1.46}$$

We now define the $(\infty, 1)$-categories of mixed-parity de Rham modules and he corresponding mixed-parity almost de Rham modules by using the objects involved to generated these categories:

Definition 13. Considering all the mixed parity de Rham bundles (locally finite free) as defined above, we consider the sub-$(\infty, 1)$-category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.47}$$

generated by the mixed-parity de Rham bundles (locally finite free ones). These are defined to be the $(\infty, 1)$-categories of mixed-parity de Rham complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.48}$$

Definition 14. Considering all the mixed parity almost de Rham bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}},\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.49}$$

generated by the mixed-parity almost de Rham bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity de Rham complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}},\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.50}$$

Then the corresponding mixed-parity de Rham functors can be extended to these categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}},\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}, \tag{1.51}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.52}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.53}$$

1.1.2 Mixed-Parity de Rham Riemann-Hilbert Correspondence

This chapter will extend the corresponding Riemann-Hilbert correspondence from [Sch1], [LZ], [BL1], [BL2], [M] to the mixed-parity setting.

Definition 15. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}, \tag{1.54}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.55}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.56}$$

to $(\infty,1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\mathrm{\acute{e}t}} \tag{1.57}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}), \tag{1.58}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}), \tag{1.59}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}), \tag{1.60}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}), \tag{1.61}$$

$$\tag{1.62}$$

respectively.

Definition 16. In the situation where we have the Frobenius action we consider the follwing. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}, \tag{1.63}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.64}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.65}$$

to $(\infty,1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\mathrm{\acute{e}t}} \tag{1.66}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}), \tag{1.67}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}), \tag{1.68}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}), \tag{1.69}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}), \tag{1.70}$$

$$\tag{1.71}$$

respectively.

1.2 Geometric Family of Mixed-Parity Hodge Modules II: Cristalline Situations

1.2.1 Period Rings and Sheaves

Reference 3. [Sch1], [KL1], [KL2], [BL1], [BL2], [BS], [BHS], [Fon1], [CS1], [CS2], [BK], [BBK], [BBBK], [KKM], [BBM], [LZ], [TT], [M].

Rings

Let X be a rigid analytic space over $\mathbb{Q}_p$. We have the corresponding étale site and the corresponding pro-étale site of X, which we denote them by $X_{\text{proét}}, X_{\text{ét}}$. The relationship of the two sites can be reflected by the corresponding morphism $f : X_{\text{proét}} \longrightarrow X_{\text{ét}}$. Then we have the corresponding cristalline period rings and sheaves from [TT]:

$$\Gamma_{\text{cristalline},X,\text{proét}}, \Gamma^{O}_{\text{cristalline},X,\text{proét}}. \tag{1.72}$$

Our notations are different from [TT], we use Γ to mean B in [TT], while Γ^{O} will be the corresponding OB ring in [TT].

Definition 17. Now we assume that $p > 2$, following [BS] we join the square root of t element in $\Gamma_{\text{cristalline},X,\text{proét}}$ which forms the sheaves:

$$\Gamma_{\text{cristalline},X,\text{proét}}\{t^{1/2}\}, \Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\}. \tag{1.73}$$

And following [BL1], [BL2], [Fon1], [BHS] we further have the following sheaves of rings:

$$\Gamma_{\text{cristalline},X,\text{proét}}\{t^{1/2}, \log(t)\}, \Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}, \log(t)\}. \tag{1.74}$$

Definition 18. We use the notations:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét}}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I} \tag{1.75}$$

to denote the perfect Robba rings from [KL1], [KL2], where $I \subset (0, \infty)$. Then we join $t^{1/2}$ to these sheaves we have:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét}}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}. \tag{1.76}$$

And following [BL1], [BL2], [Fon1], [BHS] we have the following larger sheaves:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét}}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}, \log(t)\}. \tag{1.77}$$

Definition 19. From now on, we use the same notation to denote the period rings involved tensored with a finite extension of $\mathbb{Q}_p$ containing square root of p as in [BS].

$$\Gamma_{\text{cristalline},X,\text{proét}}\{t^{1/2}\}, \Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\}. \tag{1.78}$$

$$\Gamma_{\text{cristalline},X,\text{proét}}\{t^{1/2},\log(t)\},\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2},\log(t)\}. \tag{1.79}$$

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét}}\{t^{1/2}\},\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\},\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}. \tag{1.80}$$

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét}}\{t^{1/2},\log(t)\},\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2},\log(t)\},\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2},\log(t)\}. \tag{1.81}$$

This is necessary since we to extend the action of φ to the period rings by $\varphi(t^{1/2}\otimes 1)=\varphi(t)^{1/2}\otimes 1$.

Modules

We consider quasicoherent presheaves in the following two situation:

- □ The solid quasicoherent modules from [CS1], [CS2];
- □ The ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM] with the corresponding monomorphic ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM].

Definition 20. We use the notation:

$$\text{preModule}^{\text{solid,quasicoherent}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét}}\{t^{1/2}\}},\text{preModule}^{\text{solid,quasicoherent}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}},\text{preModule}^{\text{solid,quasicoherent}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}} \tag{1.82}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Definition 21. We use the notation:

$$\text{preModule}^{\text{ind}-\text{Banach,quasicoherent}}_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét}}\{t^{1/2}\}}, \tag{1.83}$$
$$\text{preModule}^{\text{ind}-\text{Banach,quasicoherent}}_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}}, \tag{1.84}$$
$$\text{preModule}^{\text{ind}-\text{Banach,quasicoherent}}_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}} \tag{1.85}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of inductive Banach modules.

Definition 22. We use the notation:

$$\text{Module}^{\text{solid,quasicoherent}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét}}\{t^{1/2}\}},\text{Module}^{\text{solid,quasicoherent}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}},\text{Module}^{\text{solid,quasicoherent}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}} \tag{1.86}$$

to denote the $(\infty,1)$-categories of solid quasicoherent sheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Mixed-Parity Hodge Modules without Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Definition 23. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}, \tag{1.87}$$

we consider the following functor cristalline sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}) \tag{1.88}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}). \tag{1.89}$$

We call F mixed-parity cristalline if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\} \tag{1.90}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\} \tag{1.91}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\} \tag{1.92}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}. \tag{1.93}$$

Definition 24. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}, \tag{1.94}$$

we consider the following functor cristalline$^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}, \log(t)\}) \tag{1.95}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}, \log(t)\}). \tag{1.96}$$

We call F mixed-parity almost cristalline if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}, \log(t)\} \tag{1.97}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}, \log(t)\} \tag{1.98}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}, \log(t)\} \tag{1.99}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}, \log(t)\}. \tag{1.100}$$

We now define the $(\infty,1)$-categories of mixed-parity cristalline modules and he corresponding mixed-parity almost cristalline modules by using the objects involved to generated these categories:

Definition 25. Considering all the mixed parity cristalline bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.101}$$

generated by the mixed-parity cristalline bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity cristalline complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.102}$$

Definition 26. Considering all the mixed parity almost cristalline bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.103}$$

generated by the mixed-parity almost cristalline bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity cristalline complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.104}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.105}$$

Then the corresponding mixed-parity cristalline functors can be extended to these categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.106}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}, \tag{1.107}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.108}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.109}$$

Mixed-Parity Hodge Modules with Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Remark 2. All the coherent sheaves over mixed-parity Robba sheaves in this section will carry the corresponding Frobenius morphism $\varphi : F \xrightarrow{\sim} \varphi^* F$.

Definition 27. For any locally free coherent sheaf F over

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\},\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\},\tag{1.110}$$

we consider the following functor cristalline sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\})\tag{1.111}$$

or

$$f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\}).\tag{1.112}$$

We call F mixed-parity cristalline if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\})\otimes\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\}\tag{1.113}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\}\tag{1.114}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\})\otimes\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\}\tag{1.115}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\}.\tag{1.116}$$

Definition 28. For any locally free coherent sheaf F over

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\},\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\},\tag{1.117}$$

we consider the following functor cristalline$^{\text{almost}}$ sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2},\log(t)\})\tag{1.118}$$

or

$$f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2},\log(t)\}).\tag{1.119}$$

We call F mixed-parity almost cristalline if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2},\log(t)\})\otimes\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2},\log(t)\}\tag{1.120}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2},\log(t)\}\tag{1.121}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2},\log(t)\})\otimes\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2},\log(t)\}\tag{1.122}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2},\log(t)\}.\tag{1.123}$$

We now define the $(\infty,1)$-categories of mixed-parity cristalline modules and he corresponding mixed-parity almost cristalline modules by using the objects involved to generated these categories:

Definition 29. Considering all the mixed parity cristalline bundles (locally finite free) as defined above, we consider the sub-$(\infty, 1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.124}$$

generated by the mixed-parity cristalline bundles (locally finite free ones). These are defined to be the $(\infty, 1)$-categories of mixed-parity cristalline complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.125}$$

Definition 30. Considering all the mixed parity almost cristalline bundles (locally finite free) as defined above, we consider the sub-$(\infty, 1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.126}$$

generated by the mixed-parity almost cristalline bundles (locally finite free ones). These are defined to be the $(\infty, 1)$-categories of mixed-parity cristalline complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.127}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.128}$$

Then the corresponding mixed-parity cristalline functors can be extended to these categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.129}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}, \tag{1.130}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.131}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.132}$$

1.2.2 Mixed-Parity cristalline Riemann-Hilbert Correspondence

This chapter will extend the corresponding Riemann-Hilbert correspondence from [Sch1], [LZ], [BL1], [BL2], [M] to the mixed-parity setting.

Definition 31. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}, \tag{1.133}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.134}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.135}$$

to $(\infty,1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\mathrm{\acute{e}t}} \tag{1.136}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}), \tag{1.137}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}), \tag{1.138}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}), \tag{1.139}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}), \tag{1.140}$$

$$\tag{1.141}$$

respectively.

Definition 32. In the situation where we have the Frobenius action we consider the follwing. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}, \tag{1.142}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.143}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.144}$$

to $(\infty,1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\mathrm{\acute{e}t}} \tag{1.145}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}), \tag{1.146}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}), \tag{1.147}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}), \tag{1.148}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}), \tag{1.149}$$

$$\tag{1.150}$$

respectively.

1.3 Geometric Family of Mixed-Parity Hodge Modules III: Semi-Stable Situations

Reference 4. [Sch1], [KL1], [KL2], [BL1], [BL2], [BS], [BHS], [Fon1], [CS1], [CS2], [BK], [BBK], [BBBK], [KKM], [BBM], [LZ], [Shi], [M].

1.3.1 Period Rings and Sheaves

Rings

Let X be a rigid analytic space over $\mathbb{Q}_p$. We have the corresponding étale site and the corresponding pro-étale site of X, which we denote them by $X_{\text{proét}}, X_{\text{ét}}$. The relationship of the two sites can be reflected by the corresponding morphism $f : X_{\text{proét}} \longrightarrow X_{\text{ét}}$. Then we have the corresponding semi-stable period rings and sheaves from [Shi]:

$$\Gamma_{\text{semistable},X,\text{proét}}, \Gamma^{O}_{\text{semistable},X,\text{proét}}. \tag{1.151}$$

Our notations are different from [Shi], we use Γ to mean B in [Shi], while Γ^{O} will be the corresponding OB ring in [Shi].

Definition 33. Now we assume that $p > 2$, following [BS] we join the square root of t element in $\Gamma_{\text{semistable},X,\text{proét}}$ which forms the sheaves:

$$\Gamma_{\text{semistable},X,\text{proét}}\{t^{1/2}\}, \Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}\}. \tag{1.152}$$

And following [BL1], [BL2], [Fon1], [BHS] we further have the following sheaves of rings:

$$\Gamma_{\text{semistable},X,\text{proét}}\{t^{1/2}, \log(t)\}, \Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}, \log(t)\}. \tag{1.153}$$

Definition 34. We use the notations:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét}}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I} \tag{1.154}$$

to denote the perfect Robba rings from [KL1], [KL2], where $I \subset (0, \infty)$. Then we join $t^{1/2}$ to these sheaves we have:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét}}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}. \tag{1.155}$$

And following [BL1], [BL2], [Fon1], [BHS] we have the following larger sheaves:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét}}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}, \log(t)\}. \tag{1.156}$$

Definition 35. From now on, we use the same notation to denote the period rings involved tensored with a finite extension of $\mathbb{Q}_p$ containing square root of p as in [BS].

$$\Gamma_{\text{semistable},X,\text{proét}}\{t^{1/2}\}, \Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}\}. \tag{1.157}$$

$$\Gamma_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\},\Gamma^{O}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}. \tag{1.158}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}. \tag{1.159}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2},\log(t)\}. \tag{1.160}$$

This is necessary since we to extend the action of φ to the period rings by $\varphi(t^{1/2}\otimes 1)=\varphi(t)^{1/2}\otimes 1$.

Modules

We consider quasicoherent presheaves in the following two situation:

- □ The solid quasicoherent modules from [CS1], [CS2];
- □ The ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM] with the corresponding monomorphic ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM].

Definition 36. We use the notation:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.161}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Definition 37. We use the notation:

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}}, \tag{1.162}$$
$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.163}$$
$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.164}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of inductive Banach modules.

Definition 38. We use the notation:

$$\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.165}$$

to denote the $(\infty,1)$-categories of solid quasicoherent sheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Mixed-Parity Hodge Modules without Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Definition 39. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}, \tag{1.166}$$

we consider the following functor semistable sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}) \tag{1.167}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}). \tag{1.168}$$

We call F mixed-parity semi-stable if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\} \tag{1.169}$$

$$\overset{\sim}{\longrightarrow} F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\} \tag{1.170}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\} \tag{1.171}$$

$$\overset{\sim}{\longrightarrow} F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}. \tag{1.172}$$

Definition 40. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}, \tag{1.173}$$

we consider the following functor semistable$^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}) \tag{1.174}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}). \tag{1.175}$$

We call F mixed-parity almost semi-stable if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\} \tag{1.176}$$

$$\overset{\sim}{\longrightarrow} F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\} \tag{1.177}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\} \tag{1.178}$$

$$\overset{\sim}{\longrightarrow} F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}. \tag{1.179}$$

We now define the $(\infty,1)$-categories of mixed-parity semi-stable modules and he corresponding mixed-parity almost semi-stable modules by using the objects involved to generated these categories:

Definition 41. Considering all the mixed parity semi-stable bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.180}$$

generated by the mixed-parity semi-stable bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity semi-stable complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.181}$$

Definition 42. Considering all the mixed parity almost semi-stable bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.182}$$

generated by the mixed-parity almost semi-stable bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity semi-stable complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.183}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.184}$$

Then the corresponding mixed-parity semi-stable functors can be extended to these categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.185}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}, \tag{1.186}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.187}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.188}$$

Mixed-Parity Hodge Modules with Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Remark 3. All the coherent sheaves over mixed-parity Robba sheaves in this section will carry the corresponding Frobenius morphism $\varphi : F \overset{\sim}{\longrightarrow} \varphi^* F$.

Definition 43. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}, \tag{1.189}$$

we consider the following functor semistable sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}) \tag{1.190}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}). \tag{1.191}$$

We call F mixed-parity semi-stable if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\} \tag{1.192}$$

$$\overset{\sim}{\longrightarrow} F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\} \tag{1.193}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\} \tag{1.194}$$

$$\overset{\sim}{\longrightarrow} F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}. \tag{1.195}$$

Definition 44. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}, \tag{1.196}$$

we consider the following functor semistable$^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}) \tag{1.197}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}). \tag{1.198}$$

We call F mixed-parity almost semi-stable if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\} \tag{1.199}$$

$$\overset{\sim}{\longrightarrow} F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\} \tag{1.200}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\} \tag{1.201}$$

$$\overset{\sim}{\longrightarrow} F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}. \tag{1.202}$$

We now define the $(\infty,1)$-categories of mixed-parity semi-stable modules and he corresponding mixed-parity almost semi-stable modules by using the objects involved to generated these categories:

Definition 45. Considering all the mixed parity semi-stable bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}},\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \quad (1.203)$$

generated by the mixed-parity semi-stable bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity semi-stable complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}},\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \quad (1.204)$$

Definition 46. Considering all the mixed parity almost semi-stable bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}},\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \quad (1.205)$$

generated by the mixed-parity almost semi-stable bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity semi-stable complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \quad (1.206)$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \quad (1.207)$$

Then the corresponding mixed-parity semi-stable functors can be extended to these categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}},\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}, \quad (1.208)$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \quad (1.209)$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \quad (1.210)$$

1.3.2 Mixed-Parity semi-stable Riemann-Hilbert Correspondence

This chapter will extend the corresponding Riemann-Hilbert correspondence from [Sch1], [LZ], [BL1], [BL2], [M] to the mixed-parity setting.

Definition 47. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\mathrm{preModule}^{\text{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\text{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}}, \quad (1.211)$$

and

$$\mathrm{preModule}^{\text{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}}, \quad (1.212)$$

$$\mathrm{preModule}^{\text{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}} \quad (1.213)$$

to $(\infty, 1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\text{ét}} \quad (1.214)$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{semistable},X,\text{proét}}\{t^{1/2}\}), \quad (1.215)$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{semistable},X,\text{proét}}\{t^{1/2}\}), \quad (1.216)$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{semistable},X,\text{proét}}\{t^{1/2}, \log(t)\}), \quad (1.217)$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{semistable},X,\text{proét}}\{t^{1/2}, \log(t)\}), \quad (1.218)$$

$$(1.219)$$

respectively.

Definition 48. In the situation where we have the Frobenius action we consider the follwing. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\varphi\mathrm{preModule}^{\text{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\text{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}}, \quad (1.220)$$

and

$$\varphi\mathrm{preModule}^{\text{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}}, \quad (1.221)$$

$$\varphi\mathrm{preModule}^{\text{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}} \quad (1.222)$$

to $(\infty, 1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\text{ét}} \quad (1.223)$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{semistable},X,\text{proét}}\{t^{1/2}\}), \tag{1.224}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{semistable},X,\text{proét}}\{t^{1/2}\}), \tag{1.225}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{semistable},X,\text{proét}}\{t^{1/2}, \log(t)\}), \tag{1.226}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{semistable},X,\text{proét}}\{t^{1/2}, \log(t)\}), \tag{1.227}$$

$$\tag{1.228}$$

respectively.

1.4 Localizations

Reference 5. [AI1], [AI2], [AB1], [AB2], [Fon2], [Fon3], [Fa1].

1.4.1 Extension of Fundamental Groups

In the local setting setting in fact we can have more thorough understanding of more structures. Locally we can have the Galois group of $\mathbb{Q}_p \langle T_1, ..., T_n \rangle$ for some $n > 0$ in the smooth situation for instance. Our current discussion will be in the following situation:

Definition 49. We define the corresponding two fold covering of the Galois group:

$$\mathrm{Gal}(\overline{\mathbb{Q}_p \langle T_1, ..., T_n \rangle}^{\wedge} / \mathbb{Q}_p \langle T_1, ..., T_n \rangle)_2 \tag{1.229}$$

by taking the product of

$$\mathrm{Gal}(\overline{\mathbb{Q}_p \langle T_1, ..., T_n \rangle}^{\wedge} / \mathbb{Q}_p \langle T_1, ..., T_n \rangle), \mathrm{Gal}(\overline{Q}_p / \mathbb{Q}_p)_2 \tag{1.230}$$

where the latter is the group defined in [BS, Just before Lemma 7.5]. This group admits an action on the element $t^{1/2}$ through the action of the group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)_2$.

1.4.2 Modules

We consider the following definition of modules with $(\varphi, \mathrm{Gal}(\overline{\mathbb{Q}_p \langle T_1, ..., T_n \rangle}^{\wedge} / \mathbb{Q}_p \langle T_1, ..., T_n \rangle)_2)$-structure.

Definition 50. Let $R := \mathbb{Q}_p \langle T_1, ..., T_n \rangle$. We use the notation:

$$\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\}}, \mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.231}$$

to denote the $(\infty, 1)$-categories of solid modules over the corresonding Robba rings in the local setting namey associated to:

$$R^{\mathrm{perf}\flat} := \mathbb{Q}_p(p^{1/p^\infty}) \left\langle T_1^{1/p^\infty}, ..., T_n^{1/p^\infty} \right\rangle^{\wedge\flat}. \tag{1.232}$$

Then we consider all the modules as such carrying commuting operations from φ and

$$\Sigma := \mathrm{Gal}(\overline{\mathbb{Q}_p \langle T_1, ..., T_n \rangle}^{\wedge} / \mathbb{Q}_p \langle T_1, ..., T_n \rangle)_2, \tag{1.233}$$

which is assumed to be semilinear. We use the notation

$$\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\varphi,\Sigma \ \square, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\}}, \mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\varphi,\Sigma \ \square, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\varphi,\Sigma \ \square, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.234}$$

to denote the categories.

Definition 51. For any module F over

$$\Gamma^{\text{perfect}}_{\text{Robba},R,\text{proét},\infty}\{t^{1/2}\},\Gamma^{\text{perfect}}_{\text{Robba},R,\text{proét},I}\{t^{1/2}\},\tag{1.235}$$

carrying the structure of (φ,Σ)-action, we consider the following functor dR sending F to the following object:

$$(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},R,\text{proét},\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{deRham},R,\text{proét}}\{t^{1/2}\})^{\Sigma}\tag{1.236}$$

or

$$f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},R,\text{proét},I}\{t^{1/2}\}}\Gamma^{O}_{\text{deRham},R,\text{proét}}\{t^{1/2}\}).\tag{1.237}$$

We call F mixed-parity de Rham if we have the following isomorphism:

$$(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},R,\text{proét},\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{deRham},R,\text{proét}}\{t^{1/2}\})^{\Sigma}\otimes\Gamma^{O}_{\text{deRham},R,\text{proét}}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\text{deRham},R,\text{proét}}\{t^{1/2}\}\tag{1.238}$$

or

$$(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},R,\text{proét},I}\{t^{1/2}\}}\Gamma^{O}_{\text{deRham},R,\text{proét}}\{t^{1/2}\})^{\Sigma}\otimes\Gamma^{O}_{\text{deRham},R,\text{proét}}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\text{deRham},R,\text{proét}}\{t^{1/2}\}.\tag{1.239}$$

Definition 52. For any module F over

$$\Gamma^{\text{perfect}}_{\text{Robba},R,\text{proét},\infty}\{t^{1/2}\},\Gamma^{\text{perfect}}_{\text{Robba},R,\text{proét},I}\{t^{1/2}\},\tag{1.240}$$

carrying the structure of (φ,Σ)-action, we consider the following functor dR sending F to the following object:

$$(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},R,\text{proét},\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{deRham},R,\text{proét}}\{t^{1/2},\log(t)\})^{\Sigma}\tag{1.241}$$

or

$$f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},R,\text{proét},I}\{t^{1/2}\}}\Gamma^{O}_{\text{deRham},R,\text{proét}}\{t^{1/2}\}).\tag{1.242}$$

We call F mixed-parity almost de Rham if we have the following isomorphism:

$$(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},R,\text{proét},\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{deRham},R,\text{proét}}\{t^{1/2},\log(t)\})^{\Sigma}\otimes\Gamma^{O}_{\text{deRham},R,\text{proét}}\{t^{1/2},\log(t)\}\tag{1.243}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\text{deRham},R,\text{proét}}\{t^{1/2},\log(t)\}\tag{1.244}$$

or

$$(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},R,\text{proét},I}\{t^{1/2}\}}\Gamma^{O}_{\text{deRham},R,\text{proét}}\{t^{1/2},\log(t)\})^{\Sigma}\otimes\Gamma^{O}_{\text{deRham},R,\text{proét}}\{t^{1/2},\log(t)\}\tag{1.245}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\text{deRham},R,\text{proét}}\{t^{1/2},\log(t)\}.\tag{1.246}$$

Definition 53. For any module F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}, \tag{1.247}$$

carrying the structure of (φ,Σ)-action, we consider the following functor cristalline sending F to the following object:

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\})^{\Sigma} \tag{1.248}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\}). \tag{1.249}$$

We call F mixed-parity cristalline if we have the following isomorphism:

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\})^{\Sigma}\otimes\Gamma^{O}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\} \tag{1.250}$$

or

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\})^{\Sigma}\otimes\Gamma^{O}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\}. \tag{1.251}$$

Definition 54. For any module F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}, \tag{1.252}$$

carrying the structure of (φ,Σ)-action, we consider the following functor cristalline$^{\mathrm{almost}}$ sending F to the following object:

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\})^{\Sigma} \tag{1.253}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\}). \tag{1.254}$$

We call F mixed-parity almost cristalline if we have the following isomorphism:

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\})^{\Sigma}\otimes\Gamma^{O}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\} \tag{1.255}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\} \tag{1.256}$$

or

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\})^{\Sigma}\otimes\Gamma^{O}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\} \tag{1.257}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}. \tag{1.258}$$

Definition 55. For any module F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\},\tag{1.259}$$

carrying the structure of (φ,Σ)-action, we consider the following functor semistable sending F to the following object:

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\})^{\Sigma}\tag{1.260}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{deRham},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\}).\tag{1.261}$$

We call F mixed-parity semi-stable if we have the following isomorphism:

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\})^{\Sigma}\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\}\tag{1.262}$$

or

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\})^{\Sigma}\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\}.\tag{1.263}$$

Definition 56. For any module F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\},\tag{1.264}$$

carrying the structure of (φ,Σ)-action, we consider the following functor semistable$^{\mathrm{almost}}$ sending F to the following object:

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\})^{\Sigma}\tag{1.265}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\}).\tag{1.266}$$

We call F mixed-parity almost de Rham if we have the following isomorphism:

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\})^{\Sigma}\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}\tag{1.267}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}\tag{1.268}$$

or

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\})^{\Sigma}\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}\tag{1.269}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}.\tag{1.270}$$

Chapter 2

Mixed-Parity p-adic Hodge Modules in v-Topology

2.1 Geometric Family of Mixed-Parity Hodge Modules I: de Rham Situations

Reference 6. [Sch1], [Sch2], [FS], [KL1], [KL2], [BL1], [BL2], [BS], [BHS], [Fon1], [CS1], [CS2], [BK], [BBK], [BBBK], [KKM], [BBM], [LZ], [M].

2.1.1 Period Rings and Sheaves

Rings

Let X be a rigid analytic space over $\mathbb{Q}_p$. We have the corresponding étale site and the corresponding pro-étale site of X, which we denote them by $X_v, X_{\text{ét}}$. The relationship of the two sites can be reflected by the corresponding morphism $f : X_v \longrightarrow X_{\text{ét}}$. Then we have the corresponding de Rham period rings and sheaves from [Sch1]:

$$\Gamma_{\text{deRham},X,v}, \Gamma^{O}_{\text{deRham},X,v}. \tag{2.1}$$

Our notations are different from [Sch1], we use Γ to mean B in [Sch1], while Γ^O will be the corresponding OB ring in [Sch1].

Definition 57. Now we assume that $p > 2$, following [BS] we join the square root of t element in $\Gamma_{\text{deRham},X,v}$ which forms the sheaves:

$$\Gamma_{\text{deRham},X,v}\{t^{1/2}\}, \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}\}. \tag{2.2}$$

And following [BL1], [BL2], [Fon1], [BHS] we further have the following sheaves of rings:

$$\Gamma_{\text{deRham},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\}. \tag{2.3}$$

Definition 58. We use the notations:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I} \tag{2.4}$$

to denote the perfect Robba rings from [KL1], [KL2], where $I \subset (0, \infty)$. Then we join $t^{1/2}$ to these sheaves we have:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}. \tag{2.5}$$

And following [BL1], [BL2], [Fon1], [BHS] we have the following larger sheaves:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}, \log(t)\}. \tag{2.6}$$

Definition 59. From now on, we use the same notation to denote the period rings involved tensored with a finite extension of $\mathbb{Q}_p$ containing square root of p as in [BS].

$$\Gamma_{\text{deRham},X,v}\{t^{1/2}\}, \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}\}. \tag{2.7}$$

$$\Gamma_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\},\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\}. \tag{2.8}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}. \tag{2.9}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2},\log(t)\}. \tag{2.10}$$

This is necessary since we to extend the action of φ to the period rings by $\varphi(t^{1/2}\otimes 1)=\varphi(t)^{1/2}\otimes 1$.

Modules

We consider quasicoherent presheaves in the following two situation:

- □ The solid quasicoherent modules from [CS1], [CS2];
- □ The ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM] with the corresponding monomorphic ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM].

Definition 60. We use the notation:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.11}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Definition 61. We use the notation:

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}}, \tag{2.12}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.13}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.14}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of inductive Banach modules.

Definition 62. We use the notation:

$$\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.15}$$

to denote the $(\infty,1)$-categories of solid quasicoherent sheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Mixed-Parity Hodge Modules without Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Definition 63. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{2.16}$$

we consider the following functor dR sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}) \tag{2.17}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}). \tag{2.18}$$

We call F mixed-parity de Rham if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}) \otimes \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\} \overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\} \tag{2.19}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}) \otimes \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\} \overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}. \tag{2.20}$$

Definition 64. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{2.21}$$

we consider the following functor $\mathrm{dR}^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\}) \tag{2.22}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\}). \tag{2.23}$$

We call F mixed-parity almost de Rham if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\} \tag{2.24}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\} \tag{2.25}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\} \tag{2.26}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\}. \tag{2.27}$$

We now define the $(\infty,1)$-categories of mixed-parity de Rham modules and he corresponding mixed-parity almost de Rham modules by using the objects involved to generated these categories:

Definition 65. Considering all the mixed parity de Rham bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.28}$$

generated by the mixed-parity de Rham bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity de Rham complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.29}$$

Definition 66. Considering all the mixed parity almost de Rham bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.30}$$

generated by the mixed-parity almost de Rham bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity de Rham complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.31}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.32}$$

Then the corresponding mixed-parity de Rham functors can be extended to these categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.33}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{2.34}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.35}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.36}$$

Mixed-Parity Hodge Modules with Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Remark 4. All the coherent sheaves over mixed-parity Robba sheaves in this section will carry the corresponding Frobenius morphism $\varphi : F \overset{\sim}{\longrightarrow} \varphi^* F$.

Definition 67. For any locally free coherent sheaf F over

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}, \tag{2.37}$$

we consider the following functor dR sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}\}) \tag{2.38}$$

or

$$f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}\}). \tag{2.39}$$

We call F mixed-parity de Rham if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}\}) \otimes \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}\} \overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}\} \tag{2.40}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}\}) \otimes \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}\} \overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}\}. \tag{2.41}$$

Definition 68. For any locally free coherent sheaf F over

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}, \tag{2.42}$$

we consider the following functor $\text{dR}^{\text{almost}}$ sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\}) \tag{2.43}$$

or

$$f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\}). \tag{2.44}$$

We call F mixed-parity almost de Rham if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\} \tag{2.45}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\} \tag{2.46}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\} \tag{2.47}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\}. \tag{2.48}$$

We now define the $(\infty, 1)$-categories of mixed-parity de Rham modules and he corresponding mixed-parity almost de Rham modules by using the objects involved to generated these categories:

Definition 69. Considering all the mixed parity de Rham bundles (locally finite free) as defined above, we consider the sub-$(\infty, 1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.49}$$

generated by the mixed-parity de Rham bundles (locally finite free ones). These are defined to be the $(\infty, 1)$-categories of mixed-parity de Rham complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.50}$$

Definition 70. Considering all the mixed parity almost de Rham bundles (locally finite free) as defined above, we consider the sub-$(\infty, 1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.51}$$

generated by the mixed-parity almost de Rham bundles (locally finite free ones). These are defined to be the $(\infty, 1)$-categories of mixed-parity de Rham complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.52}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.53}$$

Then the corresponding mixed-parity de Rham functors can be extended to these categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{2.54}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.55}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.56}$$

2.1.2 Mixed-Parity de Rham Riemann-Hilbert Correspondence

This chapter will extend the corresponding Riemann-Hilbert correspondence from [Sch1], [LZ], [BL1], [BL2], [M] to the mixed-parity setting.

Definition 71. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{2.57}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.58}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.59}$$

to $(\infty, 1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\text{ét}} \tag{2.60}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{deRham},X,v}\{t^{1/2}\}), \tag{2.61}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{deRham},X,v}\{t^{1/2}\}), \tag{2.62}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\}), \tag{2.63}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\}), \tag{2.64}$$

$$\tag{2.65}$$

respectively.

Definition 72. In the situation where we have the Frobenius action we consider the follwing. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\varphi\mathrm{preModule}^{\text{solid,quasicoherent,mixed}-\text{paritydeRham}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\text{solid,quasicoherent,mixed}-\text{paritydeRham}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}}, \tag{2.66}$$

and

$$\varphi\mathrm{preModule}^{\text{solid,quasicoherent,mixed}-\text{parityalmostdeRham}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.67}$$

$$\varphi\mathrm{preModule}^{\text{solid,quasicoherent,mixed}-\text{parityalmostdeRham}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}} \tag{2.68}$$

to $(\infty, 1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\text{ét}} \tag{2.69}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{deRham},X,v}\{t^{1/2}\}), \tag{2.70}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{deRham},X,v}\{t^{1/2}\}), \tag{2.71}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\}), \tag{2.72}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\}), \tag{2.73}$$

$$\tag{2.74}$$

respectively.

2.2 Geometric Family of Mixed-Parity Hodge Modules II: Cristalline Situations

References: [Sch1], [Sch2], [FS], [KL1], [KL2], [BL1], [BL2], [BS], [BHS], [Fon1], [CS1], [CS2], [BK], [BBK], [BBBK], [KKM], [BBM], [LZ], [TT], [M].

2.2.1 Period Rings and Sheaves

Rings

Let X be a rigid analytic space over $\mathbb{Q}_p$. We have the corresponding étale site and the corresponding pro-étale site of X, which we denote them by $X_v, X_{\text{ét}}$. The relationship of the two sites can be reflected by the corresponding morphism $f : X_v \longrightarrow X_{\text{ét}}$. Then we have the corresponding cristalline period rings and sheaves from [TT]:

$$\Gamma_{\text{cristalline},X,v}, \Gamma^{O}_{\text{cristalline},X,v}. \tag{2.75}$$

Our notations are different from [TT], we use Γ to mean B in [TT], while Γ^{O} will be the corresponding OB ring in [TT].

Definition 73. Now we assume that $p > 2$, following [BS] we join the square root of t element in $\Gamma_{\text{cristalline},X,v}$ which forms the sheaves:

$$\Gamma_{\text{cristalline},X,v}\{t^{1/2}\}, \Gamma^{O}_{\text{cristalline},X,v}\{t^{1/2}\}. \tag{2.76}$$

And following [BL1], [BL2], [Fon1], [BHS] we further have the following sheaves of rings:

$$\Gamma_{\text{cristalline},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{O}_{\text{cristalline},X,v}\{t^{1/2}, \log(t)\}. \tag{2.77}$$

Definition 74. We use the notations:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I} \tag{2.78}$$

to denote the perfect Robba rings from [KL1], [KL2], where $I \subset (0, \infty)$. Then we join $t^{1/2}$ to these sheaves we have:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}. \tag{2.79}$$

And following [BL1], [BL2], [Fon1], [BHS] we have the following larger sheaves:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}, \log(t)\}. \tag{2.80}$$

Definition 75. From now on, we use the same notation to denote the period rings involved tensored with a finite extension of $\mathbb{Q}_p$ containing square root of p as in [BS].

$$\Gamma_{\text{cristalline},X,v}\{t^{1/2}\}, \Gamma^{O}_{\text{cristalline},X,v}\{t^{1/2}\}. \tag{2.81}$$

$$\Gamma_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\},\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}. \tag{2.82}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}. \tag{2.83}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2},\log(t)\}. \tag{2.84}$$

This is necessary since we to extend the action of φ to the period rings by $\varphi(t^{1/2}\otimes 1)=\varphi(t)^{1/2}\otimes 1$.

Modules

We consider quasicoherent presheaves in the following two situation:

- □ The solid quasicoherent modules from [CS1], [CS2];
- □ The ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM] with the corresponding monomorphic ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM].

Definition 76. We use the notation:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.85}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Definition 77. We use the notation:

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}}, \tag{2.86}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.87}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.88}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of inductive Banach modules.

Definition 78. We use the notation:

$$\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.89}$$

to denote the $(\infty,1)$-categories of solid quasicoherent sheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Mixed-Parity Hodge Modules without Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Definition 79. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{2.90}$$

we consider the following functor dR sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}) \tag{2.91}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}). \tag{2.92}$$

We call F mixed-parity cristalline if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\})\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\} \tag{2.93}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\})\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}. \tag{2.94}$$

Definition 80. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{2.95}$$

we consider the following functor $\mathrm{dR}^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}) \tag{2.96}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}). \tag{2.97}$$

We call F mixed-parity almost cristalline if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\})\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\} \tag{2.98}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\} \tag{2.99}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\})\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\} \tag{2.100}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}. \tag{2.101}$$

We now define the $(\infty,1)$-categories of mixed-parity cristalline modules and he corresponding mixed-parity almost cristalline modules by using the objects involved to generated these categories:

Definition 81. Considering all the mixed parity cristalline bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.102}$$

generated by the mixed-parity cristalline bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity cristalline complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.103}$$

Definition 82. Considering all the mixed parity almost cristalline bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.104}$$

generated by the mixed-parity almost cristalline bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity cristalline complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.105}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.106}$$

Then the corresponding mixed-parity cristalline functors can be extended to these categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.107}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{2.108}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.109}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.110}$$

Mixed-Parity Hodge Modules with Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Remark 5. All the coherent sheaves over mixed-parity Robba sheaves in this section will carry the corresponding Frobenius morphism $\varphi : F \overset{\sim}{\longrightarrow} \varphi^*F$.

Definition 83. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\},\tag{2.111}$$

we consider the following functor dR sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\})\tag{2.112}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}).\tag{2.113}$$

We call F mixed-parity cristalline if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\})\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}\tag{2.114}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\})\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}.\tag{2.115}$$

Definition 84. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\},\tag{2.116}$$

we consider the following functor $\mathrm{dR}^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\})\tag{2.117}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}).\tag{2.118}$$

We call F mixed-parity almost cristalline if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\})\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}\tag{2.119}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}\tag{2.120}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\})\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}\tag{2.121}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}.\tag{2.122}$$

We now define the $(\infty,1)$-categories of mixed-parity cristalline modules and he corresponding mixed-parity almost cristalline modules by using the objects involved to generated these categories:

Definition 85. Considering all the mixed parity cristalline bundles (locally finite free) as defined above, we consider the sub-$(\infty, 1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.123}$$

generated by the mixed-parity cristalline bundles (locally finite free ones). These are defined to be the $(\infty, 1)$-categories of mixed-parity cristalline complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.124}$$

Definition 86. Considering all the mixed parity almost cristalline bundles (locally finite free) as defined above, we consider the sub-$(\infty, 1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.125}$$

generated by the mixed-parity almost cristalline bundles (locally finite free ones). These are defined to be the $(\infty, 1)$-categories of mixed-parity cristalline complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.126}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.127}$$

Then the corresponding mixed-parity cristalline functors can be extended to these categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{2.128}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.129}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.130}$$

2.2.2 Mixed-Parity Cristalline Riemann-Hilbert Correspondence

This chapter will extend the corresponding Riemann-Hilbert correspondence from [Sch1], [LZ], [BL1], [BL2], [M] to the mixed-parity setting.

Definition 87. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{2.131}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.132}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.133}$$

to $(\infty, 1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\text{ét}} \tag{2.134}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}), \tag{2.135}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}), \tag{2.136}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}, \log(t)\}), \tag{2.137}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}, \log(t)\}), \tag{2.138}$$

$$\tag{2.139}$$

respectively.

Definition 88. In the situation where we have the Frobenius action we consider the follwing. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{2.140}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.141}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.142}$$

to $(\infty, 1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\text{ét}} \tag{2.143}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}), \tag{2.144}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}), \tag{2.145}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}, \log(t)\}), \tag{2.146}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}, \log(t)\}), \tag{2.147}$$

$$\tag{2.148}$$

respectively.

2.3 Geometric Family of Mixed-Parity Hodge Modules III: Semi-Stable Situations

References: [Sch1], [Sch2], [FS], [KL1], [KL2], [BL1], [BL2], [BS], [BHS], [Fon1], [CS1], [CS2], [BK], [BBK], [BBBK], [KKM], [BBM], [LZ], [Shi], [M].

2.3.1 Period Rings and Sheaves

Rings

Let X be a rigid analytic space over $\mathbb{Q}_p$. We have the corresponding étale site and the corresponding pro-étale site of X, which we denote them by $X_v, X_{\text{ét}}$. The relationship of the two sites can be reflected by the corresponding morphism $f : X_v \longrightarrow X_{\text{ét}}$. Then we have the corresponding semi-stable period rings and sheaves from [Shi]:

$$\Gamma_{\text{semistable},X,v}, \Gamma^{O}_{\text{semistable},X,v}. \tag{2.149}$$

Our notations are different from [Shi], we use Γ to mean B in [Shi], while Γ^O will be the corresponding OB ring in [Shi].

Definition 89. Now we assume that $p > 2$, following [BS] we join the square root of t element in $\Gamma_{\text{semistable},X,v}$ which forms the sheaves:

$$\Gamma_{\text{semistable},X,v}\{t^{1/2}\}, \Gamma^{O}_{\text{semistable},X,v}\{t^{1/2}\}. \tag{2.150}$$

And following [BL1], [BL2], [Fon1], [BHS] we further have the following sheaves of rings:

$$\Gamma_{\text{semistable},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{O}_{\text{semistable},X,v}\{t^{1/2}, \log(t)\}. \tag{2.151}$$

Definition 90. We use the notations:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I} \tag{2.152}$$

to denote the perfect Robba rings from [KL1], [KL2], where $I \subset (0, \infty)$. Then we join $t^{1/2}$ to these sheaves we have:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}. \tag{2.153}$$

And following [BL1], [BL2], [Fon1], [BHS] we have the following larger sheaves:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}, \log(t)\}. \tag{2.154}$$

Definition 91. From now on, we use the same notation to denote the period rings involved tensored with a finite extension of $\mathbb{Q}_p$ containing square root of p as in [BS].

$$\Gamma_{\text{semistable},X,v}\{t^{1/2}\}, \Gamma^{O}_{\text{semistable},X,v}\{t^{1/2}\}. \tag{2.155}$$

$$\Gamma_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\},\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}. \tag{2.156}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}. \tag{2.157}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2},\log(t)\}. \tag{2.158}$$

This is necessary since we to extend the action of φ to the period rings by $\varphi(t^{1/2}\otimes 1)=\varphi(t)^{1/2}\otimes 1$.

Modules

We consider quasicoherent presheaves in the following two situation:

- □ The solid quasicoherent modules from [CS1], [CS2];
- □ The ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM] with the corresponding monomorphic ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM].

Definition 92. We use the notation:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.159}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Definition 93. We use the notation:

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}}, \tag{2.160}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.161}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.162}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of inductive Banach modules.

Definition 94. We use the notation:

$$\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.163}$$

to denote the $(\infty,1)$-categories of solid quasicoherent sheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Mixed-Parity Hodge Modules without Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Definition 95. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{2.164}$$

we consider the following functor dR sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}) \tag{2.165}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}). \tag{2.166}$$

We call F mixed-parity semi-stable if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\} \xrightarrow{\sim} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\} \tag{2.167}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\} \xrightarrow{\sim} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}. \tag{2.168}$$

Definition 96. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{2.169}$$

we consider the following functor $\mathrm{dR}^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}, \log(t)\}) \tag{2.170}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}, \log(t)\}). \tag{2.171}$$

We call F mixed-parity almost semi-stable if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}, \log(t)\} \tag{2.172}$$

$$\xrightarrow{\sim} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}, \log(t)\} \tag{2.173}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}, \log(t)\} \tag{2.174}$$

$$\xrightarrow{\sim} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}, \log(t)\}. \tag{2.175}$$

We now define the $(\infty, 1)$-categories of mixed-parity semi-stable modules and he corresponding mixed-parity almost semi-stable modules by using the objects involved to generated these categories:

Definition 97. Considering all the mixed parity semi-stable bundles (locally finite free) as defined above, we consider the sub-$(\infty, 1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.176}$$

generated by the mixed-parity semi-stable bundles (locally finite free ones). These are defined to be the $(\infty, 1)$-categories of mixed-parity semi-stable complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.177}$$

Definition 98. Considering all the mixed parity almost semi-stable bundles (locally finite free) as defined above, we consider the sub-$(\infty, 1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.178}$$

generated by the mixed-parity almost semi-stable bundles (locally finite free ones). These are defined to be the $(\infty, 1)$-categories of mixed-parity semi-stable complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.179}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.180}$$

Then the corresponding mixed-parity semi-stable functors can be extended to these categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{2.181}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.182}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.183}$$

Mixed-Parity Hodge Modules with Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Remark 6. All the coherent sheaves over mixed-parity Robba sheaves in this section will carry the corresponding Frobenius morphism $\varphi : F \overset{\sim}{\longrightarrow} \varphi^* F$.

Definition 99. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{2.184}$$

we consider the following functor dR sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}) \tag{2.185}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}). \tag{2.186}$$

We call F mixed-parity semi-stable if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\} \overset{\sim}{\longrightarrow} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\} \tag{2.187}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\} \overset{\sim}{\longrightarrow} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}. \tag{2.188}$$

Definition 100. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{2.189}$$

we consider the following functor $\mathrm{dR}^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}, \log(t)\}) \tag{2.190}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}, \log(t)\}). \tag{2.191}$$

We call F mixed-parity almost semi-stable if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}, \log(t)\} \tag{2.192}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}, \log(t)\} \tag{2.193}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}, \log(t)\} \tag{2.194}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}, \log(t)\}. \tag{2.195}$$

We now define the $(\infty, 1)$-categories of mixed-parity semi-stable modules and he corresponding mixed-parity almost semi-stable modules by using the objects involved to generated these categories:

Definition 101. Considering all the mixed parity semi-stable bundles (locally finite free) as defined above, we consider the sub-$(\infty, 1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.196}$$

generated by the mixed-parity semi-stable bundles (locally finite free ones). These are defined to be the $(\infty, 1)$-categories of mixed-parity semi-stable complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.197}$$

Definition 102. Considering all the mixed parity almost semi-stable bundles (locally finite free) as defined above, we consider the sub-$(\infty, 1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.198}$$

generated by the mixed-parity almost semi-stable bundles (locally finite free ones). These are defined to be the $(\infty, 1)$-categories of mixed-parity semi-stable complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.199}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.200}$$

Then the corresponding mixed-parity semi-stable functors can be extended to these categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{2.201}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.202}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.203}$$

2.3.2 Mixed-Parity semi-stable Riemann-Hilbert Correspondence

This chapter will extend the corresponding Riemann-Hilbert correspondence from [Sch1], [LZ], [BL1], [BL2], [M] to the mixed-parity setting.

Definition 103. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{2.204}$$

and

$$\text{preModule}^{\text{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.205}$$

$$\text{preModule}^{\text{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}} \tag{2.206}$$

to $(\infty,1)$-categories in image denoted by:

$$\text{preModule}_{X,\text{ét}} \tag{2.207}$$

to be the following functors sending each F in the domain to:

$$\text{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{semistable},X,v}\{t^{1/2}\}), \tag{2.208}$$

$$\text{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\text{semistable},X,v}\{t^{1/2}\}), \tag{2.209}$$

$$\text{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{semistable},X,v}\{t^{1/2},\log(t)\}), \tag{2.210}$$

$$\text{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\text{semistable},X,v}\{t^{1/2},\log(t)\}), \tag{2.211}$$

$$\tag{2.212}$$

respectively.

Definition 104. In the situation where we have the Frobenius action we consider the follwing. We define the following Riemann-Hilbert functor $\text{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\varphi\text{preModule}^{\text{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\text{preModule}^{\text{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}}, \tag{2.213}$$

and

$$\varphi\text{preModule}^{\text{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.214}$$

$$\varphi\text{preModule}^{\text{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}} \tag{2.215}$$

to $(\infty,1)$-categories in image denoted by:

$$\text{preModule}_{X,\text{ét}} \tag{2.216}$$

to be the following functors sending each F in the domain to:

$$\text{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{semistable},X,v}\{t^{1/2}\}), \tag{2.217}$$

$$\text{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\text{semistable},X,v}\{t^{1/2}\}), \tag{2.218}$$

$$\text{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{semistable},X,v}\{t^{1/2},\log(t)\}), \tag{2.219}$$

$$\text{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\text{semistable},X,v}\{t^{1/2},\log(t)\}), \tag{2.220}$$

$$\tag{2.221}$$

respectively.

Remark 7. We now have discussed the corresponding two different morphisms:

$$f : X_{\text{proét}} \longrightarrow X_{\text{ét}}; \tag{2.222}$$
$$f' : X_{\text{v}} \longrightarrow X_{\text{ét}}. \tag{2.223}$$

One can consider the following relation among the sites:

$$X_{\text{v}} \longrightarrow X_{\text{proét}} \longrightarrow X_{\text{ét}} \tag{2.224}$$

which produces f'. The map:

$$g : X_{\text{v}} \longrightarrow X_{\text{proét}} \tag{2.225}$$

can help us relate the corresponding constructions above as in [B, Proposition 2.37]. Namely we have:

$$\mathrm{dR}_{v} = \mathrm{dR}_{\text{proét}} g_*; \tag{2.226}$$
$$\mathrm{dR}_{v,\text{almost}} = \mathrm{dR}_{\text{proét,almost}} g_*; \tag{2.227}$$
$$\text{cristalline}_{v} = \text{cristalline}_{\text{proét}} g_*; \tag{2.228}$$
$$\text{cristalline}_{v,\text{almost}} = \text{cristalline}_{\text{proét,almost}} g_*; \tag{2.229}$$
$$\text{semistable}_{v} = \text{semistable}_{\text{proét}} g_*; \tag{2.230}$$
$$\text{semistable}_{v,\text{almost}} = \text{semistable}_{\text{proét,almost}} g_*. \tag{2.231}$$

Chapter 3

Mixed-Parity Hodge Modules over v-Stacks

3.1 $(\infty,1)$-Quasicoherent Sheaves over Extended Fargues-Fontaine Curves I

We now consider the sheaves over extended Fargues-Fontain stacks:

Remark 8. Let X be a general small v-stack over $\mathbb{Q}_p$ (as a v-stack[1]).

Definition 105.

$$\mathrm{FF}_X := \bigcup_{I\subset(0,\infty)} \mathrm{Spa}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I\subset(0,\infty)}\{t^{1/2}\}\otimes_{\mathbb{Q}_p} E, \Gamma^{\mathrm{perfect},+}_{\mathrm{Robba},X,I\subset(0,\infty)}\{t^{1/2}\}\otimes_{\mathbb{Q}_p} E)/\varphi^{\mathbb{Z}}, \tag{3.1}$$

which has the corresonding structure map as in the following:

$$\begin{array}{c}\mathrm{FF}_X\\ \big\downarrow\\ \mathrm{FF}_{\mathrm{Spd}^\diamond(\mathbb{Q}_p)}.\end{array}$$

Definition 106. We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} \tag{3.2}$$

to denote $(\infty,1)$-category of all the solid quasicoherent sheaves over the stack FF_X. For any local perfectoid $Y\in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} \tag{3.3}$$

to denote $(\infty,1)$-category of all the solid quasicoherent sheaves over the stack FF_X which are perfect complexes. For any local perfectoid $Y\in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 107. We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} \tag{3.4}$$

to denote $(\infty,1)$-category of all the ind-Banach quasicoherent sheaves over the stack FF_X. For any local perfectoid $Y\in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} \tag{3.5}$$

to denote $(\infty,1)$-category of all the indBanach quasicoherent sheaves over the stack FF_X which are perfect complexes. For any local perfectoid $Y\in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.

[1] All v-stacks in this chapter are assumed to be over a v-stack associated to $\mathbb{Q}_p$ like this.

Definition 108. We use the notation

$$\{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)} \tag{3.6}$$

to denote $(\infty,1)$-category of all the solid φ-modules over the extended Robba ring. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I\subset J\subset K$. For any local perfectoid $Y\in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\{\underset{\mathrm{solid,perfectcomplexes}}{\varphi\mathrm{Module}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)} \tag{3.7}$$

to denote $(\infty,1)$-category of all the solid φ-modules over the extended Robba ring which are perfect complexes. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I\subset J\subset K$. For any local perfectoid $Y\in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 109. We use the notation

$$\{\varphi\mathrm{Module}^{\mathrm{indBanach}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)} \tag{3.8}$$

to denote $(\infty,1)$-category of all the ind-Banach φ-modules over the extended Robba ring. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I\subset J\subset K$. For any local perfectoid $Y\in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\left\{\underset{\mathrm{indBanach,perfectcomplexes},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E}{\varphi\mathrm{Module}}\right\}_{I\subset(0,\infty)} \tag{3.9}$$

to denote $(\infty,1)$-category of all the ind-Banach φ-modules over the extended Robba ring which are perfect complexes. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I\subset J\subset K$. For any local perfectoid $Y\in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.

Proposition 1. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{array}{ccc}
\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} & \longrightarrow & \{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)} \\
\downarrow & & \downarrow \\
\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} & \longrightarrow & \{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}.
\end{array}$$

Proposition 2. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$
\begin{array}{ccc}
\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} & \longrightarrow & \{\underset{\mathrm{solid,perfectcomplexes}}{\varphi\mathrm{Module}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p} E)\}_{I\subset(0,\infty)} \\
\downarrow & & \downarrow \\
\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} & \longrightarrow & \{\underset{\mathrm{solid,perfectcomplexes}}{\varphi\mathrm{Module}}(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p} E)\}_{I\subset(0,\infty)}.
\end{array}
$$

Proposition 3. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$
\begin{array}{ccc}
\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} & \longrightarrow & \{\varphi\mathrm{Module}^{indBanach}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p} E)\}_{I\subset(0,\infty)} \\
\downarrow & & \downarrow \\
\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} & \longrightarrow & \{\varphi\mathrm{Module}^{\mathrm{indBanach}}(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p} E)\}_{I\subset(0,\infty)}.
\end{array}
$$

Proposition 4. *We have the following commutative diagram by taking the global section functor*

in the horizontal rows:

$$
\begin{array}{ccc}
\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} & \rightarrow & \left\{ \begin{array}{c} \varphi\mathrm{Module} \\ \mathrm{indBanach,perfectcomplexes},\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E \end{array} \right\}_{I\subset(0,\infty)} \\
\downarrow & & \downarrow \\
\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} & \longrightarrow & \left\{ \begin{array}{c} \varphi\mathrm{Module} \\ \mathrm{indBanach,perfectcomplexes},\Gamma^{perfect}_{Robba,X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E \end{array} \right\}_{I\subset(0,\infty)}
\end{array}.
$$

Taking the corresponding simplicial commutative object we have the following propositions:

Proposition 5. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$
\begin{array}{ccc}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} & \longrightarrow & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \\
\downarrow & & \downarrow \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}}} & \longrightarrow & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}}
\end{array}.
$$

Proposition 6. *We have the following commutative diagram by taking the global section functor*

in the horizontal rows:

$$
\begin{array}{ccc}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} & \longrightarrow & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid,perfectcomplexes}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \\
\downarrow & & \downarrow \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}}} & \longrightarrow & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid,perfectcomplexes}}(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}}
\end{array}.
$$

Proposition 7. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$
\begin{array}{ccc}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}}} & \longrightarrow & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{indBanach}}(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \\
\uparrow & & \uparrow \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} & \longrightarrow & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{indBanach}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}}
\end{array}.
$$

Proposition 8. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$
\begin{array}{ccc}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}}} & \longrightarrow & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{indBanach,perfectcomplexes}}(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \\
\uparrow & & \uparrow \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} & \longrightarrow & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{indBanach,perfectcomplexes}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}}
\end{array}.
$$

3.2 $(\infty,1)$-Quasicoherent Sheaves over Extended Fargues-Fontaine Curves II

We now consider the sheaves over extended Fargues-Fontain stacks:

Remark 9. Let X be a general small v-stack over $\mathbb{Q}_p$ (as a v-stack[2]). Spa will denote Clausen-Scholze analytic space in [CS2].

Definition 110.

$$\mathrm{FF}_X := \bigcup_{I\subset(0,\infty)} \mathrm{Spa}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I\subset(0,\infty)}\{t^{1/2},\log(t)\} \otimes_{\mathbb{Q}_p} E, \Gamma^{\mathrm{perfect},+}_{\mathrm{Robba},X,I\subset(0,\infty)}\{t^{1/2},\log(t)\} \otimes_{\mathbb{Q}_p} E)/\varphi^{\mathbb{Z}}, \tag{3.10}$$

[3]which has the corresonding structure map as in the following:

$$\begin{array}{c} \mathrm{FF}_X \\ \big\downarrow \\ \mathrm{FF}_{\mathrm{Spd}^\diamond(\mathbb{Q}_p)}. \end{array}$$

Definition 111. We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} \tag{3.12}$$

to denote $(\infty,1)$-category of all the solid quasicoherent sheaves over the stack FF_X. For any local perfectoid $Y \in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} \tag{3.13}$$

to denote $(\infty,1)$-category of all the solid quasicoherent sheaves over the stack FF_X which are perfect complexes. For any local perfectoid $Y \in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 112. We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} \tag{3.14}$$

---

[2]All v-stacks in this chapter are assumed to be over a v-stack associated to $\mathbb{Q}_p$ like this.

[3]Here the ring $\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I\subset(0,\infty)}\{t^{1/2},\log(t)\}$ is defined to be just:

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I\subset(0,\infty)}\{t^{1/2}\}[\log(t)] \tag{3.11}$$

which carries the corresponding adic topology from the corresponding Banach ring $\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I\subset(0,\infty)}\{t^{1/2}\}$, which induces a topological adic ring structure (therefore a corresponding condensed animated ring structure in [CS2]). Then the corresponding spectrum will be defined to be the correponding analytic spectrum from Clausen-Scholze.

to denote $(\infty,1)$-category of all the ind-Banach quasicoherent sheaves over the stack FF_X. For any local perfectoid $Y \in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} \tag{3.15}$$

to denote $(\infty,1)$-category of all the indBanach quasicoherent sheaves over the stack FF_X which are perfect complexes. For any local perfectoid $Y \in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 113. We use the notation

$$\{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I}\{t^{1/2},\log(t)\} \otimes_{\mathbb{Q}_p} E)\}_{I\subset(0,\infty)} \tag{3.16}$$

to denote $(\infty,1)$-category of all the solid φ-modules over the extended Robba ring. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I \subset J \subset K$. For any local perfectoid $Y \in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\left\{ \underset{\mathrm{solid,perfectcomplexes},(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p} E)}{\varphi\mathrm{Module}} \right\}_{I\subset(0,\infty)} \tag{3.17}$$

to denote $(\infty,1)$-category of all the solid φ-modules over the extended Robba ring which are perfect complexes. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I \subset J \subset K$. For any local perfectoid $Y \in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 114. We use the notation

$$\{\underset{\mathrm{indBanach}}{\varphi\mathrm{Module}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I}\{t^{1/2},\log(t)\} \otimes_{\mathbb{Q}_p} E)\}_{I\subset(0,\infty)} \tag{3.18}$$

to denote $(\infty,1)$-category of all the ind-Banach φ-modules over the extended Robba ring. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I \subset J \subset K$. For any local perfectoid $Y \in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\left\{ \underset{\mathrm{indBanach,perfectcomplexes},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p} E}{\varphi\mathrm{Module}} \right\}_{I\subset(0,\infty)} \tag{3.19}$$

to denote $(\infty,1)$-category of all the ind-Banach φ-modules over the extended Robba ring which are perfect complexes. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I \subset J \subset K$. For any local perfectoid $Y \in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.

Proposition 9. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$
\begin{array}{ccc}
\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} & \longrightarrow & \{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)} \\
\downarrow & & \downarrow \\
\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} & \longrightarrow & \{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}.
\end{array}
$$

Proposition 10. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$
\begin{array}{ccc}
\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} & \longrightarrow & \left\{\underset{\mathrm{solid,perfectcomplexes},(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)}{\varphi\mathrm{Module}}\right\}_{I\subset(0,\infty)} \\
\downarrow & & \downarrow \\
\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} & \longrightarrow & \left\{\underset{\mathrm{solid,perfectcomplexes},(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)}{\varphi\mathrm{Module}}\right\}_{I\subset(0,\infty)}.
\end{array}
$$

Proposition 11. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$
\begin{array}{ccc}
\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} & \longrightarrow & \{\underset{\mathrm{indBanach}}{\varphi\mathrm{Module}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)} \\
\downarrow & & \downarrow \\
\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} & \longrightarrow & \{\underset{\mathrm{indBanach}}{\varphi\mathrm{Module}}(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}.
\end{array}
$$

Proposition 12. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{CD}
\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} @>>> \left\{\begin{matrix}\varphi\mathrm{Module}\\ \mathrm{indBanach,perfectcomplexes},\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E\end{matrix}\right\}_{I\subset(0,\infty)}\\
@VVV @VVV\\
\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} @>>> \left\{\begin{matrix}\varphi\mathrm{Module}\\ \mathrm{indBanach,perfectcomplexes},\Gamma^{perfect}_{Robba,X,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E\end{matrix}\right\}_{I\subset(0,\infty)}
\end{CD}.$$

Taking the corresponding simplicial commutative object we have the following propositions:

Proposition 13. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{CD}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} @>>> \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}}\\
@VVV @VVV\\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}}} @>>> \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}}
\end{CD}.$$

Proposition 14. *We have the following commutative diagram by taking the global section functor*

in the horizontal rows:

$$\begin{array}{ccc}
\text{SimplicialRings}_{\text{Quasicoherent}^{\text{solid,perfectcomplexes}}_{\text{FF}_{\text{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\text{FF}_{\text{Spd}(\mathbb{Q}_p)^\diamond}}}} & \longrightarrow & \text{SimplicialRings}_{\{\varphi\text{Module}^{\text{solid,perfectcomplexes}}(\Gamma^{perfect}_{Robba,\text{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \\
\downarrow & & \downarrow \\
\text{SimplicialRings}_{\text{Quasicoherent}^{\text{solid,perfectcomplexes}}_{\text{FF}_X,\mathcal{O}_{\text{FF}_X}}} & \longrightarrow & \text{SimplicialRings}_{\{\varphi\text{Module}^{\text{solid,perfectcomplexes}}(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}}
\end{array}.$$

Proposition 15. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{array}{ccc}
\text{SimplicialRings}_{\text{Quasicoherent}^{\text{indBanach}}_{\text{FF}_X,\mathcal{O}_{\text{FF}_X}}} & \longrightarrow & \text{SimplicialRings}_{\{\varphi\text{Module}^{\text{indBanach}}(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \\
\uparrow & & \uparrow \\
\text{SimplicialRings}_{\text{Quasicoherent}^{\text{indBanach}}_{\text{FF}_{\text{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\text{FF}_{\text{Spd}(\mathbb{Q}_p)^\diamond}}}} & \longrightarrow & \text{SimplicialRings}_{\{\varphi\text{Module}^{indBanach}(\Gamma^{perfect}_{Robba,\text{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}}
\end{array}.$$

Proposition 16. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{array}{ccc}
\text{SimplicialRings}_{\text{Quasicoherent}^{\text{indBanach,perfectcomplexes}}_{\text{FF}_X,\mathcal{O}_{\text{FF}_X}}} & \longrightarrow & \text{SimplicialRings}_{\{\varphi\text{Module}^{\text{indBanach,perfectcomplexes}}(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \\
\uparrow & & \uparrow \\
\text{SimplicialRings}_{\text{Quasicoherent}^{\text{indBanach,perfectcomplexes}}_{\text{FF}_{\text{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\text{FF}_{\text{Spd}(\mathbb{Q}_p)^\diamond}}}} & \longrightarrow & \text{SimplicialRings}_{\{\varphi\text{Module}^{\text{indBanach,perfectcomplexes}}(\Gamma^{perfect}_{Robba,\text{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}}
\end{array}.$$

3.3 Geometric Family of Mixed-Parity Hodge Modules I: de Rham Situations

Reference 7. [Sch1], [Sch2], [FS], [KL1], [KL2], [BL1], [BL2], [BS], [BHS], [Fon1], [CS1], [CS2], [BK], [BBK], [BBBK], [KKM], [BBM], [LZ], [M].

3.3.1 Period Rings and Sheaves

Rings

Let X be a v-stack over $\mathrm{Spd}\mathbb{Q}_p$, which is required to be restricted to be a diamond which is further assumed to be spacial in the local setting. We have the corresponding étale site and the corresponding pro-étale site of X, which we denote them by X_v, $X_{\text{ét}}$. The relationship of the two sites can be reflected by the corresponding morphism $f : X_v \longrightarrow X_{\text{ét}}$. Then we have the corresponding de Rham period rings and sheaves from [Sch1]:

$$\Gamma_{\mathrm{deRham},X,v}, \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}. \tag{3.20}$$

Our notations are different from [Sch1], we use Γ to mean B in [Sch1], while $\Gamma^{\mathcal{O}}$ will be the corresponding $\mathcal{O}B$ ring in [Sch1].

Definition 115. Now we assume that $p > 2$, following [BS] we join the square root of t element in $\Gamma_{\mathrm{deRham},X,v}$ which forms the sheaves:

$$\Gamma_{\mathrm{deRham},X,v}\{t^{1/2}\}, \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2}\}. \tag{3.21}$$

And following [BL1], [BL2], [Fon1], [BHS] we further have the following sheaves of rings:

$$\Gamma_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\}. \tag{3.22}$$

Definition 116. We use the notations:

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I} \tag{3.23}$$

to denote the perfect Robba rings from [KL1], [KL2], where $I \subset (0,\infty)$. Then we join $t^{1/2}$ to these sheaves we have:

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}. \tag{3.24}$$

And following [BL1], [BL2], [Fon1], [BHS] we have the following larger sheaves:

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}, \log(t)\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}, \log(t)\}. \tag{3.25}$$

Definition 117. From now on, we use the same notation to denote the period rings involved tensored with a finite extension of $\mathbb{Q}_p$ containing square root of p as in [BS].

$$\Gamma_{\mathrm{deRham},X,v}\{t^{1/2}\}, \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2}\}. \tag{3.26}$$

$$\Gamma_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\},\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\}. \tag{3.27}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}. \tag{3.28}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2},\log(t)\}. \tag{3.29}$$

This is necessary since we to extend the action of φ to the period rings by $\varphi(t^{1/2}\otimes 1)=\varphi(t)^{1/2}\otimes 1$.

Modules

We consider quasicoherent presheaves in the following two situation:

- The solid quasicoherent modules from [CS1], [CS2];
- The ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM] with the corresponding monomorphic ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM].

Definition 118. We use the notation:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.30}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Definition 119. We use the notation:

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}}, \tag{3.31}$$
$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.32}$$
$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.33}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of inductive Banach modules.

Definition 120. We use the notation:

$$\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.34}$$

to denote the $(\infty,1)$-categories of solid quasicoherent sheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Mixed-Parity Hodge Modules without Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Definition 121. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\},\tag{3.35}$$

we consider the following functor dR sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\})\tag{3.36}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}).\tag{3.37}$$

We call F mixed-parity de Rham if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\})\otimes\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}\tag{3.38}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\})\otimes\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}.\tag{3.39}$$

Definition 122. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\},\tag{3.40}$$

we consider the following functor $\mathrm{dR}^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\})\tag{3.41}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\}).\tag{3.42}$$

We call F mixed-parity almost de Rham if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\})\otimes\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\}\tag{3.43}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\}\tag{3.44}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\})\otimes\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\}\tag{3.45}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\}.\tag{3.46}$$

We now define the $(\infty,1)$-categories of mixed-parity de Rham modules and he corresponding mixed-parity almost de Rham modules by using the objects involved to generated these categories:

Definition 123. Considering all the mixed parity de Rham bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.47}$$

generated by the mixed-parity de Rham bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity de Rham complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.48}$$

Definition 124. Considering all the mixed parity almost de Rham bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.49}$$

generated by the mixed-parity almost de Rham bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity de Rham complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.50}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.51}$$

Then the corresponding mixed-parity de Rham functors can be extended to these categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.52}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{3.53}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.54}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.55}$$

Mixed-Parity Hodge Modules with Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Remark 10. All the coherent sheaves over mixed-parity Robba sheaves in this section will carry the corresponding Frobenius morphism $\varphi : F \overset{\sim}{\longrightarrow} \varphi^* F$.

Definition 125. For any locally free coherent sheaf F over

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}, \tag{3.56}$$

we consider the following functor dR sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}\}) \tag{3.57}$$

or

$$f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}\}). \tag{3.58}$$

We call F mixed-parity de Rham if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}\}) \otimes \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}\} \overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}\} \tag{3.59}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}\}) \otimes \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}\} \overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}\}. \tag{3.60}$$

Definition 126. For any locally free coherent sheaf F over

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}, \tag{3.61}$$

we consider the following functor $\text{dR}^{\text{almost}}$ sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\}) \tag{3.62}$$

or

$$f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\}). \tag{3.63}$$

We call F mixed-parity almost de Rham if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\} \tag{3.64}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\} \tag{3.65}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\} \tag{3.66}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\}. \tag{3.67}$$

We now define the $(\infty, 1)$-categories of mixed-parity de Rham modules and he corresponding mixed-parity almost de Rham modules by using the objects involved to generated these categories:

Definition 127. Considering all the mixed parity de Rham bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.68}$$

generated by the mixed-parity de Rham bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity de Rham complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.69}$$

Definition 128. Considering all the mixed parity almost de Rham bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.70}$$

generated by the mixed-parity almost de Rham bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity de Rham complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.71}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.72}$$

Then the corresponding mixed-parity de Rham functors can be extended to these categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{3.73}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.74}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.75}$$

3.3.2 Mixed-Parity de Rham Riemann-Hilbert Correspondence

This chapter will extend the corresponding Riemann-Hilbert correspondence from [Sch1], [LZ], [BL1], [BL2], [M] to the mixed-parity setting.

Definition 129. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{3.76}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.77}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.78}$$

to $(\infty,1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\mathrm{\acute{e}t}} \tag{3.79}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2}\}), \tag{3.80}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2}\}), \tag{3.81}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\}), \tag{3.82}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\}), \tag{3.83}$$

$$\tag{3.84}$$

respectively.

Definition 130. In the situation where we have the Frobenius action we consider the follwing. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{3.85}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.86}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.87}$$

to $(\infty,1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\mathrm{\acute{e}t}} \tag{3.88}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2}\}), \tag{3.89}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2}\}), \tag{3.90}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\}), \tag{3.91}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\}), \tag{3.92}$$

$$\tag{3.93}$$

respectively.

3.4 Geometric Family of Mixed-Parity Hodge Modules II: Cristalline Situations

References: [Sch1], [Sch2], [FS], [KL1], [KL2], [BL1], [BL2], [BS], [BHS], [Fon1], [CS1], [CS2], [BK], [BBK], [BBBK], [KKM], [BBM], [LZ], [TT], [M].

3.4.1 Period Rings and Sheaves

Rings

Let X be a v-stack over $\mathrm{Spd}\mathbb{Q}_p$, which is required to be restricted to be a diamond which is further assumed to be spacial in the local setting. We have the corresponding étale site and the corresponding pro-étale site of X, which we denote them by X_v, $X_{\text{ét}}$. The relationship of the two sites can be reflected by the corresponding morphism $f : X_v \longrightarrow X_{\text{ét}}$. Then we have the corresponding cristalline period rings and sheaves from [TT]:

$$\Gamma_{\text{cristalline},X,v}, \Gamma^{O}_{\text{cristalline},X,v}. \tag{3.94}$$

Our notations are different from [TT], we use Γ to mean B in [TT], while Γ^O will be the corresponding OB ring in [TT].

Definition 131. Now we assume that $p > 2$, following [BS] we join the square root of t element in $\Gamma_{\text{cristalline},X,v}$ which forms the sheaves:

$$\Gamma_{\text{cristalline},X,v}\{t^{1/2}\}, \Gamma^{O}_{\text{cristalline},X,v}\{t^{1/2}\}. \tag{3.95}$$

And following [BL1], [BL2], [Fon1], [BHS] we further have the following sheaves of rings:

$$\Gamma_{\text{cristalline},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{O}_{\text{cristalline},X,v}\{t^{1/2}, \log(t)\}. \tag{3.96}$$

Definition 132. We use the notations:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I} \tag{3.97}$$

to denote the perfect Robba rings from [KL1], [KL2], where $I \subset (0, \infty)$. Then we join $t^{1/2}$ to these sheaves we have:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}. \tag{3.98}$$

And following [BL1], [BL2], [Fon1], [BHS] we have the following larger sheaves:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}, \log(t)\}. \tag{3.99}$$

Definition 133. From now on, we use the same notation to denote the period rings involved tensored with a finite extension of $\mathbb{Q}_p$ containing square root of p as in [BS].

$$\Gamma_{\text{cristalline},X,v}\{t^{1/2}\}, \Gamma^{O}_{\text{cristalline},X,v}\{t^{1/2}\}. \tag{3.100}$$

$$\Gamma_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\},\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}. \tag{3.101}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}. \tag{3.102}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2},\log(t)\}. \tag{3.103}$$

This is necessary since we to extend the action of φ to the period rings by $\varphi(t^{1/2}\otimes 1)=\varphi(t)^{1/2}\otimes 1$.

Modules

We consider quasicoherent presheaves in the following two situation:

- □ The solid quasicoherent modules from [CS1], [CS2];
- □ The ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM] with the corresponding monomorphic ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM].

Definition 134. We use the notation:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.104}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Definition 135. We use the notation:

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}}, \tag{3.105}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.106}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.107}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of inductive Banach modules.

Definition 136. We use the notation:

$$\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.108}$$

to denote the $(\infty,1)$-categories of solid quasicoherent sheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Mixed-Parity Hodge Modules without Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Definition 137. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\},\tag{3.109}$$

we consider the following functor dR sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2}\})\tag{3.110}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2}\}).\tag{3.111}$$

We call F mixed-parity cristalline if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2}\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2}\}\tag{3.112}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2}\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2}\}.\tag{3.113}$$

Definition 138. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\},\tag{3.114}$$

we consider the following functor $\mathrm{dR}^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\})\tag{3.115}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}).\tag{3.116}$$

We call F mixed-parity almost cristalline if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}\tag{3.117}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}\tag{3.118}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}\tag{3.119}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}.\tag{3.120}$$

We now define the $(\infty,1)$-categories of mixed-parity cristalline modules and he corresponding mixed-parity almost cristalline modules by using the objects involved to generated these categories:

Definition 139. Considering all the mixed parity cristalline bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.121}$$

generated by the mixed-parity cristalline bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity cristalline complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.122}$$

Definition 140. Considering all the mixed parity almost cristalline bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.123}$$

generated by the mixed-parity almost cristalline bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity cristalline complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.124}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.125}$$

Then the corresponding mixed-parity cristalline functors can be extended to these categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.126}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{3.127}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.128}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.129}$$

Mixed-Parity Hodge Modules with Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Remark 11. All the coherent sheaves over mixed-parity Robba sheaves in this section will carry the corresponding Frobenius morphism $\varphi : F \overset{\sim}{\longrightarrow} \varphi^* F$.

Definition 141. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\},\tag{3.130}$$

we consider the following functor dR sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\})\tag{3.131}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}).\tag{3.132}$$

We call F mixed-parity cristalline if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\})\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}\tag{3.133}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\})\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}.\tag{3.134}$$

Definition 142. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\},\tag{3.135}$$

we consider the following functor $\mathrm{dR}^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\})\tag{3.136}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}).\tag{3.137}$$

We call F mixed-parity almost cristalline if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\})\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}\tag{3.138}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}\tag{3.139}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\})\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}\tag{3.140}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}.\tag{3.141}$$

We now define the $(\infty,1)$-categories of mixed-parity cristalline modules and he corresponding mixed-parity almost cristalline modules by using the objects involved to generated these categories:

Definition 143. Considering all the mixed parity cristalline bundles (locally finite free) as defined above, we consider the sub-$(\infty, 1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.142}$$

generated by the mixed-parity cristalline bundles (locally finite free ones). These are defined to be the $(\infty, 1)$-categories of mixed-parity cristalline complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.143}$$

Definition 144. Considering all the mixed parity almost cristalline bundles (locally finite free) as defined above, we consider the sub-$(\infty, 1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.144}$$

generated by the mixed-parity almost cristalline bundles (locally finite free ones). These are defined to be the $(\infty, 1)$-categories of mixed-parity cristalline complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.145}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.146}$$

Then the corresponding mixed-parity cristalline functors can be extended to these categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{3.147}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.148}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.149}$$

3.4.2 Mixed-Parity Cristalline Riemann-Hilbert Correspondence

This chapter will extend the corresponding Riemann-Hilbert correspondence from [Sch1], [LZ], [BL1], [BL2], [M] to the mixed-parity setting.

Definition 145. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{3.150}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.151}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.152}$$

to $(\infty,1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\text{ét}} \tag{3.153}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}), \tag{3.154}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}), \tag{3.155}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}), \tag{3.156}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}), \tag{3.157}$$

(3.158)

respectively.

Definition 146. In the situation where we have the Frobenius action we consider the follwing. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{3.159}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.160}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.161}$$

to $(\infty,1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\text{ét}} \tag{3.162}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}), \tag{3.163}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}), \tag{3.164}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}), \tag{3.165}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}), \tag{3.166}$$

(3.167)

respectively.

3.5 Geometric Family of Mixed-Parity Hodge Modules III: Semi-Stable Situations

References: [Sch1], [Sch2], [FS], [KL1], [KL2], [BL1], [BL2], [BS], [BHS], [Fon1], [CS1], [CS2], [BK], [BBK], [BBBK], [KKM], [BBM], [LZ], [Shi], [M].

3.5.1 Period Rings and Sheaves

Rings

Let X be a v-stack over $\mathrm{Spd}\mathbb{Q}_p$, which is required to be restricted to be a diamond which is further assumed to be spacial in the local setting. We have the corresponding étale site and the corresponding pro-étale site of X, which we denote them by $X_v, X_{\text{ét}}$. The relationship of the two sites can be reflected by the corresponding morphism $f: X_v \longrightarrow X_{\text{ét}}$. Then we have the corresponding semi-stable period rings and sheaves from [Shi]:

$$\Gamma_{\text{semistable},X,v}, \Gamma^{O}_{\text{semistable},X,v}. \tag{3.168}$$

Our notations are different from [Shi], we use Γ to mean B in [Shi], while Γ^O will be the corresponding OB ring in [Shi].

Definition 147. Now we assume that $p > 2$, following [BS] we join the square root of t element in $\Gamma_{\text{semistable},X,v}$ which forms the sheaves:

$$\Gamma_{\text{semistable},X,v}\{t^{1/2}\}, \Gamma^{O}_{\text{semistable},X,v}\{t^{1/2}\}. \tag{3.169}$$

And following [BL1], [BL2], [Fon1], [BHS] we further have the following sheaves of rings:

$$\Gamma_{\text{semistable},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{O}_{\text{semistable},X,v}\{t^{1/2}, \log(t)\}. \tag{3.170}$$

Definition 148. We use the notations:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I} \tag{3.171}$$

to denote the perfect Robba rings from [KL1], [KL2], where $I \subset (0, \infty)$. Then we join $t^{1/2}$ to these sheaves we have:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}. \tag{3.172}$$

And following [BL1], [BL2], [Fon1], [BHS] we have the following larger sheaves:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}, \log(t)\}. \tag{3.173}$$

Definition 149. From now on, we use the same notation to denote the period rings involved tensored with a finite extension of $\mathbb{Q}_p$ containing square root of p as in [BS].

$$\Gamma_{\text{semistable},X,v}\{t^{1/2}\}, \Gamma^{O}_{\text{semistable},X,v}\{t^{1/2}\}. \tag{3.174}$$

$$\Gamma_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\},\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}. \tag{3.175}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}. \tag{3.176}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2},\log(t)\}. \tag{3.177}$$

This is necessary since we to extend the action of φ to the period rings by $\varphi(t^{1/2}\otimes 1)=\varphi(t)^{1/2}\otimes 1$.

Modules

We consider quasicoherent presheaves in the following two situation:

- The solid quasicoherent modules from [CS1], [CS2];
- The ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM] with the corresponding monomorphic ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM].

Definition 150. We use the notation:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.178}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Definition 151. We use the notation:

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}}, \tag{3.179}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.180}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.181}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of inductive Banach modules.

Definition 152. We use the notation:

$$\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.182}$$

to denote the $(\infty,1)$-categories of solid quasicoherent sheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Mixed-Parity Hodge Modules without Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Definition 153. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{3.183}$$

we consider the following functor dR sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}) \tag{3.184}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}). \tag{3.185}$$

We call F mixed-parity semi-stable if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\} \tag{3.186}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}. \tag{3.187}$$

Definition 154. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{3.188}$$

we consider the following functor $\mathrm{dR}^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}) \tag{3.189}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}). \tag{3.190}$$

We call F mixed-parity almost semi-stable if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\} \tag{3.191}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\} \tag{3.192}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\} \tag{3.193}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}. \tag{3.194}$$

We now define the $(\infty,1)$-categories of mixed-parity semi-stable modules and he corresponding mixed-parity almost semi-stable modules by using the objects involved to generated these categories:

Definition 155. Considering all the mixed parity semi-stable bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.195}$$

generated by the mixed-parity semi-stable bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity semi-stable complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.196}$$

Definition 156. Considering all the mixed parity almost semi-stable bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.197}$$

generated by the mixed-parity almost semi-stable bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity semi-stable complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.198}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.199}$$

Then the corresponding mixed-parity semi-stable functors can be extended to these categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{3.200}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.201}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.202}$$

Mixed-Parity Hodge Modules with Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Remark 12. All the coherent sheaves over mixed-parity Robba sheaves in this section will carry the corresponding Frobenius morphism $\varphi : F \overset{\sim}{\longrightarrow} \varphi^* F$.

Definition 157. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\},\tag{3.203}$$

we consider the following functor dR sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2}\})\tag{3.204}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2}\}).\tag{3.205}$$

We call F mixed-parity semi-stable if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2}\})\otimes\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2}\}\tag{3.206}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2}\})\otimes\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2}\}.\tag{3.207}$$

Definition 158. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\},\tag{3.208}$$

we consider the following functor $\mathrm{dR}^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\})\tag{3.209}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}).\tag{3.210}$$

We call F mixed-parity almost semi-stable if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\})\otimes\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}\tag{3.211}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}\tag{3.212}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\})\otimes\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}\tag{3.213}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}.\tag{3.214}$$

We now define the $(\infty,1)$-categories of mixed-parity semi-stable modules and he corresponding mixed-parity almost semi-stable modules by using the objects involved to generated these categories:

Definition 159. Considering all the mixed parity semi-stable bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.215}$$

generated by the mixed-parity semi-stable bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity semi-stable complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.216}$$

Definition 160. Considering all the mixed parity almost semi-stable bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.217}$$

generated by the mixed-parity almost semi-stable bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity semi-stable complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.218}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.219}$$

Then the corresponding mixed-parity semi-stable functors can be extended to these categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{3.220}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.221}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.222}$$

3.5.2 Mixed-Parity semi-stable Riemann-Hilbert Correspondence

This chapter will extend the corresponding Riemann-Hilbert correspondence from [Sch1], [LZ], [BL1], [BL2], [M] to the mixed-parity setting.

Definition 161. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{3.223}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.224}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.225}$$

to $(\infty,1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\mathrm{ét}} \tag{3.226}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}), \tag{3.227}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}), \tag{3.228}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}), \tag{3.229}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}), \tag{3.230}$$

$$\tag{3.231}$$

respectively.

Definition 162. In the situation where we have the Frobenius action we consider the follwing. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{3.232}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.233}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.234}$$

to $(\infty,1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\mathrm{ét}} \tag{3.235}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}), \tag{3.236}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}), \tag{3.237}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}), \tag{3.238}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}), \tag{3.239}$$

$$\tag{3.240}$$

respectively.

Remark 13. We now have discussed the corresponding two different morphisms:

$$f : X_{\text{proét}} \longrightarrow X_{\text{ét}}; \tag{3.241}$$
$$f' : X_{\text{v}} \longrightarrow X_{\text{ét}}. \tag{3.242}$$

One can consider the following relation among the sites:

$$X_{\text{v}} \longrightarrow X_{\text{proét}} \longrightarrow X_{\text{ét}} \tag{3.243}$$

which produces f'. The map:

$$g : X_{\text{v}} \longrightarrow X_{\text{proét}} \tag{3.244}$$

can help us relate the corresponding constructions above as in [B, Proposition 2.37]. Namely we have:

$$\mathrm{dR}_{v} = \mathrm{dR}_{\text{proét}} g_*; \tag{3.245}$$
$$\mathrm{dR}_{v,\text{almost}} = \mathrm{dR}_{\text{proét,almost}} g_*; \tag{3.246}$$
$$\text{cristalline}_{v} = \text{cristalline}_{\text{proét}} g_*; \tag{3.247}$$
$$\text{cristalline}_{v,\text{almost}} = \text{cristalline}_{\text{proét,almost}} g_*; \tag{3.248}$$
$$\text{semistable}_{v} = \text{semistable}_{\text{proét}} g_*; \tag{3.249}$$
$$\text{semistable}_{v,\text{almost}} = \text{semistable}_{\text{proét,almost}} g_*. \tag{3.250}$$

Chapter 4

Generalized Langlands Program

4.1 Moduli v-Stack

References: [FS], [FF], [Sch1],[Sch2], [KL1], [KL2];
Further References:[Lan1], [Drin1], [Drin2], [Zhu], [DHKM].

We consider the category of all the perfectoid spaces over $\overline{\mathbb{Q}_p((\mu_{p^\infty}))^\wedge}^{\wedge,\flat}$ as in [FS]. We use the notation Perfectoid_v to denote the associated v-site after [FS], [Sch2]. Let $p > 2$. For any $\mathrm{Spa}(A, A^+) \in \mathrm{perfectoid}_v$ we have the perfect Robba rings from [KL1], [KL2]:

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Spa}(A,A^+),I\subset(0,\infty)}. \tag{4.1}$$

We also have the corresponding de Rham period rings:

$$\Gamma^{+}_{\mathrm{deRham},\mathrm{Spa}(A,A^+)}, \Gamma_{\mathrm{deRham},\mathrm{Spa}(A,A^+)}. \tag{4.2}$$

In the first filtration of this first de Rham period ring we have the generator t, we now extend the corresponding rings above by adding the square root of t, $t^{1/2}$ following [BS]. We then have the extended rings:

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Spa}(A,A^+),I\subset(0,\infty)}\{t^{1/2}\}, \tag{4.3}$$

$$\Gamma^{+}_{\mathrm{deRham},\mathrm{Spa}(A,A^+)}\{t^{1/2}\}, \Gamma_{\mathrm{deRham},\mathrm{Spa}(A,A^+)}\{t^{1/2}\}. \tag{4.4}$$

Then we form the corresponding extended Fargues-Fontaine curve (after choosing a large finite extension E of $\mathbb{Q}_p$ containing $\varphi(t)^{1/2}$):

$$\mathrm{FF}_A := \bigcup_{I\subset(0,\infty)} \mathrm{Spa}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Spa}(A,A^+),I\subset(0,\infty)}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E, \Gamma^{\mathrm{perfect},+}_{\mathrm{Robba},\mathrm{Spa}(A,A^+),I\subset(0,\infty)}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E)/\varphi^{\mathbb{Z}}, \tag{4.5}$$

where the Frobenius is extended to $t^{1/2} \otimes 1$ by acting $\varphi(t)^{1/2} \otimes 1$.

Definition 163. Let G be any p-adic group as in [FS][1]. We now define the pre-v-stack Moduli_G to be a presheaf valued in the groupoid over

$$\mathrm{perfectoid}_v \tag{4.6}$$

sendind each $\mathrm{Spa}(A, A^+)$ perfectoid in the site to the groupoid of all the locally finite free coherent sheaves carrying G-bundle structure over

$$\mathrm{FF}_A := \bigcup_{I\subset(0,\infty)} \mathrm{Spa}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Spa}(A,A^+),I\subset(0,\infty)}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E, \Gamma^{\mathrm{perfect},+}_{\mathrm{Robba},\mathrm{Spa}(A,A^+),I\subset(0,\infty)}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E)/\varphi^{\mathbb{Z}}. \tag{4.7}$$

Proposition 17. *This prestack is a small v-stack in the v-topology.*

Proof. The proof will be the same as in [FS, Proposition III.1.3]. Our stack can also be regarded as a two components extension of the original stack in [FS]. □

[1]That is to say the group G is defined over $\mathbb{Q}_p$. And the Robba rings are defined over $\mathbb{Q}_p$ as well, which strictly speaking are generated from Witt vectors in [KL1], but one can generalize this directly to the level of [KL2] by replacing the field $\overline{\mathbb{Q}_p((\mu_{p^\infty}))^\wedge}^{\wedge,\flat}$ with some larger field $\overline{F((\mu_{p^\infty}))^\wedge}^{\wedge,\flat}$, where $F/\mathbb{Q}_p$ is finite extension of $\mathbb{Q}_p$.

4.2 Motives over Moduli_G

With the notation in the previous section, we now consider the sheaves over extended Fargues-Fontain stacks:

Definition 164.

$$\mathrm{FF}_{\mathrm{Moduli}_G} := \bigcup_{I\subset(0,\infty)} \mathrm{Spa}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Moduli}_G,I\subset(0,\infty)}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E, \Gamma^{\mathrm{perfect},+}_{\mathrm{Robba},\mathrm{Moduli}_G,I\subset(0,\infty)}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E)/\varphi^{\mathbb{Z}}, \tag{4.8}$$

which has the corresonding structure map as in the following:

$$\begin{array}{c} \mathrm{FF}_{\mathrm{Moduli}_G} \\ \downarrow \\ \mathrm{FF}_{\mathrm{FF}_*} \\ \downarrow \\ \mathrm{FF}_{\mathrm{Spd}^\diamond(\mathbb{Q}_p)} \\ \downarrow \\ \mathrm{Spd}^\diamond(\mathbb{Q}_p). \end{array}$$

Definition 165. We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} \tag{4.9}$$

to denote $(\infty,1)$-category of all the solid quasicoherent sheaves over the stack $\mathrm{FF}_{\mathrm{Moduli}_G}$. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} \tag{4.10}$$

to denote $(\infty,1)$-category of all the solid quasicoherent sheaves over the stack $\mathrm{FF}_{\mathrm{Moduli}_G}$ which are perfect complexes. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 166. We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} \tag{4.11}$$

to denote $(\infty,1)$-category of all the ind-Banach quasicoherent sheaves over the stack $\mathrm{FF}_{\mathrm{Moduli}_G}$. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local

sense.
We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} \tag{4.12}$$

to denote $(\infty,1)$-category of all the indBanach quasicoherent sheaves over the stack $\mathrm{FF}_{\mathrm{Moduli}_G}$ which are perfect complexes. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 167. We use the notation

$$\{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba,Moduli}_G,I}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E)\}_{I\subset(0,\infty)} \tag{4.13}$$

to denote $(\infty,1)$-category of all the solid φ-modules over the extended Robba ring. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I \subset J \subset K$. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\{\varphi\mathrm{Module}^{\mathrm{solid,perfectcomplexes}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba,Moduli}_G,I}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E)\}_{I\subset(0,\infty)} \tag{4.14}$$

to denote $(\infty,1)$-category of all the solid φ-modules over the extended Robba ring which are perfect complexes. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I \subset J \subset K$. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 168. We use the notation

$$\{\varphi\mathrm{Module}^{\mathrm{indBanach}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba,Moduli}_G,I}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E)\}_{I\subset(0,\infty)} \tag{4.15}$$

to denote $(\infty,1)$-category of all the ind-Banach φ-modules over the extended Robba ring. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I \subset J \subset K$. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\{\varphi\mathrm{Module}^{\mathrm{ind-Banach,perfectcomplexes}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba,Moduli}_G,I}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E)\}_{I\subset(0,\infty)} \tag{4.16}$$

to denote $(\infty,1)$-category of all the ind-Banach φ-modules over the extended Robba ring which are perfect complexes. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I \subset J \subset K$. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 169. We use the notation

$$\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba,Moduli}_G,\infty}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E) \tag{4.17}$$

to denote $(\infty,1)$-category of all the solid φ-modules over the extended Robba ring. The modules satisfy the Frobenius pullback condition. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the

corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\varphi\mathrm{Module}^{\mathrm{solid,perfectcomplexes}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Moduli}_{G},\infty}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E) \tag{4.18}$$

to denote $(\infty,1)$-category of all the solid φ-modules over the extended Robba ring which are perfect complexes. The modules satisfy the Frobenius pullback condition. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 170. We use the notation

$$\varphi\mathrm{Module}^{\mathrm{indBanach}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Moduli}_{G},\infty}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E) \tag{4.19}$$

to denote $(\infty,1)$-category of all the ind-Banach φ-modules over the extended Robba ring. The modules satisfy the Frobenius pullback condition. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\left\{\varphi\mathrm{Module}_{\mathrm{indBanach,perfectcomplexes},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E}\right\}_{I\subset(0,\infty)} \tag{4.20}$$

to denote $(\infty,1)$-category of all the ind-Banach φ-modules over the extended Robba ring which are perfect complexes. The modules satisfy the Frobenius pullback condition. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.

Proposition 18. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{array}{ccc}
\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} & \longrightarrow & \{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)} \\
\downarrow & & \downarrow \\
\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} & \longrightarrow & \{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}.
\end{array}$$

Proposition 19. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{array}{ccc}
\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} & \longrightarrow & \{\varphi\mathrm{Module}_{\mathrm{solid,perfectcomplexes}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)} \\
\downarrow & & \downarrow \\
\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} & \longrightarrow & \{\varphi\mathrm{Module}_{\mathrm{solid,perfectcomplexes}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}.
\end{array}$$

Proposition 20. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{array}{ccc}
\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} & \longrightarrow & \{\varphi\mathrm{Module}^{indBanach}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)} \\
\downarrow & & \downarrow \\
\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} & \longrightarrow & \{\varphi\mathrm{Module}^{\mathrm{indBanach}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}.
\end{array}$$

Proposition 21. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{array}{ccc}
\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} & \longrightarrow & \left\{\varphi\mathrm{Module}_{\mathrm{indBanach,perfectcomplexes},\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E}\right\}_{I\subset(0,\infty)} \\
\downarrow & & \downarrow \\
\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} & \longrightarrow & \left\{\varphi\mathrm{Module}_{\mathrm{indBanach,perfectcomplexes},\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E}\right\}_{I\subset(0,\infty)}.
\end{array}$$

Taking the corresponding simplicial commutative object we have the following propositions:

Proposition 22. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$
\begin{array}{ccc}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} & \longrightarrow & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \\
\downarrow & & \downarrow \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}}} & \longrightarrow & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}}
\end{array}.
$$

Proposition 23. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$
\begin{array}{ccc}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} & \longrightarrow & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid,perfectcomplexes}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \\
\downarrow & & \downarrow \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}}} & \longrightarrow & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid,perfectcomplexes}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}}
\end{array}.
$$

Proposition 24. *We have the following commutative diagram by taking the global section functor*

in the horizontal rows:

$$\begin{array}{ccc}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}}} & \longrightarrow & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{indBanach}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \\
\uparrow & & \uparrow \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} & \longrightarrow & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{indBanach}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}}
\end{array}.$$

Proposition 25. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{array}{ccc}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}}} & \longrightarrow & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{indBanach,perfectcomplexes}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \\
\uparrow & & \uparrow \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} & \longrightarrow & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{indBanach,perfectcomplexes}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}}
\end{array}.$$

4.3 Moduli v-Stack in More General Setting

References: [FS], [FF], [Sch1],[Sch2], [KL1], [KL2];
Further References: [Lan1], [Drin1], [Drin2], [Zhu], [DHKM].

We consider the category of all the perfectoid spaces over $\mathrm{Spd}\overline{\mathbb{F}}_p$ as in [FS]. We use the notation perfectoid$_v$ to denote the associated v-site after [FS], [Sch2]. Let $p > 2$. Now we fix a finite extension K of $\mathbb{Q}_p$. And the Robba rings are defined over K as well, namely we consider the generalized Witt vector over O_K as in [KL2][2]. For any $\mathrm{Spa}(A, A^+) \in$ perfectoid$_v$ we have the perfect Robba rings from [KL1], [KL2]:

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Spa}(A,A^+),I\subset(0,\infty)}. \tag{4.21}$$

We also have the corresponding de Rham period rings:

$$\Gamma^{+}_{\mathrm{deRham},\mathrm{Spa}(A,A^+)}, \Gamma_{\mathrm{deRham},\mathrm{Spa}(A,A^+)}. \tag{4.22}$$

In the first filtration of this first de Rham period ring we have the generator t, we now extend the corresponding rings above by adding the square root of t, $t^{1/2}$ following [BS]. We then have the extended rings:

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Spa}(A,A^+),I\subset(0,\infty)}\{t^{1/2}\}, \tag{4.23}$$

$$\Gamma^{+}_{\mathrm{deRham},\mathrm{Spa}(A,A^+)}\{t^{1/2}\}, \Gamma_{\mathrm{deRham},\mathrm{Spa}(A,A^+)}\{t^{1/2}\}. \tag{4.24}$$

Then we form the corresponding extended Fargues-Fontaine curve (after choosing a large finite extension E of $\mathbb{Q}_p$ containing $\varphi(t)^{1/2}$):

$$\mathrm{FF}_A := \bigcup_{I\subset(0,\infty)} \mathrm{Spa}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Spa}(A,A^+),I\subset(0,\infty)}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E, \Gamma^{\mathrm{perfect},+}_{\mathrm{Robba},\mathrm{Spa}(A,A^+),I\subset(0,\infty)}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E)/\varphi^{\mathbb{Z}}, \tag{4.25}$$

where the Frobenius is extended to $t^{1/2} \otimes 1$ by acting $\varphi(t)^{1/2} \otimes 1$.

Definition 171. Let G/K be any p-adic group as in [FS]. That is to say the group G is defined over K, where K is some finite extension of $\mathbb{Q}_p$, defined as above. We now define the pre-v-stack Moduli_G to be a presheaf valued in the groupoid over

$$\mathrm{perfectoid}_v \tag{4.26}$$

sendind each $\mathrm{Spa}(A, A^+)$ perfectoid in the site to the groupoid of all the locally finite free coherent sheaves carrying G-bundle structure over

$$\mathrm{FF}_A := \bigcup_{I\subset(0,\infty)} \mathrm{Spa}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Spa}(A,A^+),I\subset(0,\infty)}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E, \Gamma^{\mathrm{perfect},+}_{\mathrm{Robba},\mathrm{Spa}(A,A^+),I\subset(0,\infty)}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E)/\varphi^{\mathbb{Z}}. \tag{4.27}$$

[2] In [FS] and [KL2], this field is denoted by E where the relative p-adic Hodge theory in [KL2] and the Langlands correspondence in [FS] both happen over this field E. To be more precise relative p-adic Hodge theory studies p-adic cohomologization over analytic stacks over E, while the Langlands correspondences relates derived ∞-categories of $\mathrm{Moduli}_{G/E}$ and derived ∞-categories of moduli stack of representations of $W_{E,2}$ into the Langlands dual groups.

Proposition 26. *This prestack is a small v-stack in the v-topology.*

Proof. The proof will be the same as in [FS, Proposition III.1.3]. Our stack can also be regarded as a two components extension of the original stack in [FS]. □

4.4 Motives over Moduli_G in More General Setting

Keeping the generality in the previous section, we now consider the sheaves over extended Fargues-Fontain stacks:

Definition 172.

$$\mathrm{FF}_{\mathrm{Moduli}_G} := \bigcup_{I\subset(0,\infty)} \mathrm{Spa}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Moduli}_G,I\subset(0,\infty)}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E, \Gamma^{\mathrm{perfect},+}_{\mathrm{Robba},\mathrm{Moduli}_G,I\subset(0,\infty)}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E)/\varphi^{\mathbb{Z}}, \tag{4.28}$$

which has the corresonding structure map as in the following:

$$\begin{array}{c} \mathrm{FF}_{\mathrm{Moduli}_G} \\ \downarrow \\ \mathrm{FF}_{\mathrm{FF}_*} \\ \downarrow \\ \mathrm{FF}_{\mathrm{Spd}^\diamond(\mathbb{Q}_p)} \\ \downarrow \\ \mathrm{Spd}^\diamond(\mathbb{Q}_p). \end{array}$$

Definition 173. We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} \tag{4.29}$$

to denote $(\infty,1)$-category of all the solid quasicoherent sheaves over the stack $\mathrm{FF}_{\mathrm{Moduli}_G}$. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} \tag{4.30}$$

to denote $(\infty,1)$-category of all the solid quasicoherent sheaves over the stack $\mathrm{FF}_{\mathrm{Moduli}_G}$ which are perfect complexes. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 174. We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} \tag{4.31}$$

to denote $(\infty,1)$-category of all the ind-Banach quasicoherent sheaves over the stack $\mathrm{FF}_{\mathrm{Moduli}_G}$. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} \tag{4.32}$$

to denote $(\infty,1)$-category of all the indBanach quasicoherent sheaves over the stack $\mathrm{FF}_{\mathrm{Moduli}_G}$ which are perfect complexes. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 175. We use the notation

$$\{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Moduli}_G,I}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E)\}_{I\subset(0,\infty)} \tag{4.33}$$

to denote $(\infty,1)$-category of all the solid φ-modules over the extended Robba ring. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I \subset J \subset K$. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\{\varphi\mathrm{Module}^{\mathrm{solid,perfectcomplexes}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Moduli}_G,I}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E)\}_{I\subset(0,\infty)} \tag{4.34}$$

to denote $(\infty,1)$-category of all the solid φ-modules over the extended Robba ring which are perfect complexes. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I \subset J \subset K$. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 176. We use the notation

$$\{\varphi\mathrm{Module}^{\mathrm{indBanach}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Moduli}_G,I}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E)\}_{I\subset(0,\infty)} \tag{4.35}$$

to denote $(\infty,1)$-category of all the ind-Banach φ-modules over the extended Robba ring. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I \subset J \subset K$. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\{\varphi\mathrm{Module}^{\mathrm{ind-Banach,perfectcomplexes}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Moduli}_G,I}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E)\}_{I\subset(0,\infty)} \tag{4.36}$$

to denote $(\infty,1)$-category of all the ind-Banach φ-modules over the extended Robba ring which are perfect complexes. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I \subset J \subset K$. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 177. We use the notation

$$\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Moduli}_{G},\infty}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E) \tag{4.37}$$

to denote $(\infty,1)$-category of all the solid φ-modules over the extended Robba ring. The modules satisfy the Frobenius pullback condition. For any local perfectoid $Y\in\mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\varphi\mathrm{Module}^{\mathrm{solid,perfectcomplexes}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Moduli}_{G},\infty}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E) \tag{4.38}$$

to denote $(\infty,1)$-category of all the solid φ-modules over the extended Robba ring which are perfect complexes. The modules satisfy the Frobenius pullback condition. For any local perfectoid $Y\in\mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 178. We use the notation

$$\varphi\mathrm{Module}^{\mathrm{indBanach}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Moduli}_{G},\infty}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E) \tag{4.39}$$

to denote $(\infty,1)$-category of all the ind-Banach φ-modules over the extended Robba ring. The modules satisfy the Frobenius pullback condition. For any local perfectoid $Y\in\mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\left\{\varphi\mathrm{Module}_{\mathrm{indBanach,perfectcomplexes},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E}\right\}_{I\subset(0,\infty)} \tag{4.40}$$

to denote $(\infty,1)$-category of all the ind-Banach φ-modules over the extended Robba ring which are perfect complexes. The modules satisfy the Frobenius pullback condition. For any local perfectoid $Y\in\mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.

Proposition 27. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{array}{ccc}
\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} & \longrightarrow & \{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)} \\
\downarrow & & \downarrow \\
\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} & \longrightarrow & \{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}.
\end{array}$$

Proposition 28. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{array}{ccc}
\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} & \longrightarrow & \{\underset{\mathrm{solid,perfectcomplexes}}{\varphi\mathrm{Module}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)} \\
\downarrow & & \downarrow \\
\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} & \longrightarrow & \{\underset{\mathrm{solid,perfectcomplexes}}{\varphi\mathrm{Module}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}.
\end{array}$$

Proposition 29. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{array}{ccc}
\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} & \longrightarrow & \{\varphi\mathrm{Module}^{indBanach}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)} \\
\downarrow & & \downarrow \\
\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} & \longrightarrow & \{\varphi\mathrm{Module}^{\mathrm{indBanach}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}.
\end{array}$$

Proposition 30. *We have the following commutative diagram by taking the global section functor*

in the horizontal rows:

$$
\begin{array}{ccc}
\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} & \longrightarrow & \left\{ \varphi\mathrm{Module}_{\mathrm{indBanach,perfectcomplexes},\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E} \right\}_{I\subset(0,\infty)} \\
\downarrow & & \downarrow \\
\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} & \longrightarrow & \left\{ \varphi\mathrm{Module}_{\mathrm{indBanach,perfectcomplexes},\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E} \right\}_{I\subset(0,\infty)}
\end{array}.
$$

Taking the corresponding simplicial commutative object we have the following propositions:

Proposition 31. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$
\begin{array}{ccc}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} & \longrightarrow & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \\
\downarrow & & \downarrow \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}}} & \longrightarrow & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}}
\end{array}.
$$

Proposition 32. *We have the following commutative diagram by taking the global section functor*

in the horizontal rows:

$$\begin{array}{ccc}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} & \longrightarrow & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid,perfectcomplexes}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \\
\downarrow & & \downarrow \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}}} & \longrightarrow & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid,perfectcomplexes}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}}
\end{array}.$$

Proposition 33. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{array}{ccc}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}}} & \longrightarrow & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{indBanach}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \\
\uparrow & & \uparrow \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} & \longrightarrow & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{indBanach}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}}
\end{array}.$$

Proposition 34. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{array}{ccc}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}}} & \longrightarrow & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{indBanach,perfectcomplexes}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \\
\uparrow & & \uparrow \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} & \longrightarrow & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{indBanach,perfectcomplexes}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}}
\end{array}.$$

4.5 Generalized Local Langlands Correspondence after Fargues-Scholze

Reference 8.
□ References on p-adic Hodge Theory:
[pHodgeT], [pHodgeF], [pHodgeS1], [pHodgeS2], [pHodgeKL1], [pHodgeKL2], [pHodgeBS], [pHodgeKPX];
□ References on foundations of p-adic analysis:
[pToAnCS1], [pToAnCS2], [pToAnCS3], [pToAnCS4], [pToAnBBBK];
□ References on local Langlands program:
[LPL], [LPD1], [LPLL], [LPVL], [LPC], [LPFS], [LPGL], [LPEGH], [LPEG], [LPZ], [LPDHKM], [LPD2];
□ References on p-adic Langlands program:
[LPL], [LPD1], [LPLL], [LPVL], [LPC], [LPFS], [LPGL], [LPEGH], [LPEG], [LPZ], [LPDHKM], [LPD2].

4.5.1 Perfectoid Moduli Stacks

As above, we discussed certain mixed-parity moduli stacks over the mixed-parity Fargues-Fontaine curves where we work directly over the *tilt* of some field $K := \overline{\mathbb{Q}_p((\mu_{p^\infty}))}^\wedge$. We assume first that we still work over this field where we have the corresponding element t as $\log[\varepsilon]$ as in [pHodgeS1] for the de Rham sheaves over the pro-étale site. Then in the scenario where we have the splitting $t^{1/2}$ we then extend the underlying diamond to be a two-fold covering. To be more precise we defined the corresponding prestack

$$\mathrm{Bundle}_{2,G}(\mathrm{FF}_{2,\Box}) \tag{4.41}$$

which sends each perfectoid space $X_\Box$ in the v-site $\mathrm{Perfectoid}_{K,v}$ to the corresponding groupoid of G-torsors over the relative extended Fargues-Fontaine curves $\mathrm{FF}_{2,X_\Box}$. Here we assume G is the p-adic group in [LPFS], namely we have a base field K' such that G is defined over this field and we assume other conditions from [LPFS]. This is indeed a v-stack as in [LPFS, Chapter III, Proposition 1.3].

Proposition 35.

$$\mathrm{Bundle}_{2,G}(\mathrm{FF}_{2,\Box}) \tag{4.42}$$

is a small v-stack over the site $\mathrm{Perfectoid}_{K,v}$. □

In the corresponding discussion as above we actually condidered the p-adic motives over the corresponding v-stack here which is also *small*. Namely locally for each corresponding local perfectoid Y we have the corresponding chance to attach the corresonding Robba rings[3]

$$\widetilde{\Pi}_{Y,I,2}, \tag{4.43}$$

$$\widetilde{\Pi}_{Y,\infty,2}, \tag{4.44}$$

$$\widetilde{\Pi}_{Y,2}, \tag{4.45}$$

in the correspoding extended setting where we join the element $t^{1/2}$. We can have the corresponding Frobenius action or not actualy extended from the corresponding Robba rings. Be careful *from now on* we consider the following assumption from [pHodgeKL2] and [LPFS]:

Assumption 1. We assume that the corresponding G and the corresponding Robba rings are defined over a same field K' which is finite extension of $\mathbb{Q}_p$, where we often implicize the corresponding field here. We assume the field is p-adic which is denoted by E in [LPFS].

This will generalize the above discussion to the situation where we work over the field $\overline{K'((\mu_{p^\infty}))}^{\wedge,\flat}$ which contains the field $k_{K'}$, the residue field of K'. Then we have the corresponding definitions of the stacks as in the above after [pHodgeKL2] and [LPFS].

[3]Note that we need to tensor the Robba rings with a large finite extension of $\mathbb{Q}_p$ to guarantee that $\varphi.(t^{1/2})$ makes sense in the rings.

4.5.2 Generalization of Langlands Program and the Geometrization

Assumption 2. We assume that the corresponding G and the corresponding Robba rings are defined over a same field K' which is finite extension of $\mathbb{Q}_p$, where we often implicize the corresponding field here. We assume the field is p-adic which is denoted by E in [LPFS]. The $\mathrm{Bundle}_{2,G}$ is defined over $\mathrm{perfectoid}_{\mathrm{Spd}\overline{\mathbb{F}}_p}$.

The main conjecture in [LPFS] relates derived ∞-category of all the A-valued complexes over $\mathrm{Bundle}_G(\mathrm{FF}_\square)$ to the corresponding derived ∞-category of coherent sheaves over the stack of L-parameter $\mathrm{Stack}_{L,G^{\mathrm{dual}},A}/G^{\mathrm{dual}}$. The former derived category in our setting can be defined directly which is also well-defined from [LPFS] and [pHodgeS2]:

$$\mathrm{DerivedCat}(\mathrm{Bundle}_{2,G}(\mathrm{FF}_{2,\square}))_{\mathrm{Coeff}:A} \tag{4.46}$$

which is actually not expected to completely isomorphic to the corresponding category of the latter in our situation:

$$\mathrm{DerivedCat}_{\mathrm{bounded,coherent,Nilpotent}}(\mathrm{Stack}_{2,L,G^{\mathrm{dual}},A}/G^{\mathrm{dual}}) \tag{4.47}$$

where the stack of L-parameters (which origin two-fold covering of the Weil group $\mathrm{Weil}_{K',2}$) is the pull back of the stack $\mathrm{Stack}_{L,G^{\mathrm{dual}},A}/G^{\mathrm{dual}}$ along the covering map of $\mathrm{Weil}_{K',2}$ over $\mathrm{Weil}_{K'}$. Fargues-Scholze's conjecture conjectures directly that they are isomophic in the corresponding usual non-mixed-parity situation. In our situation the relationship should be clear as well but there is some tiny different due to the fact that we need to tensor the Robba rings with a large finite extension of $\mathbb{Q}_p$ to guarantee that $\varphi.(t^{1/2})$ makes sense in the rings.

Conjecture 1. (Fargues-Scholze, [LPFS, Chapter I, Conjecture 10.2]) There is a canonical isomorphism between the two ∞-categories:

$$\mathrm{DerivedCat}(\mathrm{Bundle}_G(\mathrm{FF}_\square))_{\mathrm{Coeff}:A} \tag{4.48}$$

with

$$\mathrm{DerivedCat}_{\mathrm{bounded,coherent,Nilpotent}}(\mathrm{Stack}_{L,G^{\mathrm{dual}},A}/G^{\mathrm{dual}}). \tag{4.49}$$

Conjecture 2. (After Fargues-Scholze, [LPFS, Chapter I, Conjecture 10.2]) There is a canonical direct relationship between the two ∞-categories:

$$\mathrm{DerivedCat}(\mathrm{Bundle}_G(\mathrm{FF}_\square))_{\mathrm{Coeff}:A} \tag{4.50}$$

with

$$\mathrm{DerivedCat}(\mathrm{Bundle}_{2,G}(\mathrm{FF}_{2,\square}))_{\mathrm{Coeff}:A} \tag{4.51}$$

after we consider the pull-back of the categories along the map:

$$\mathrm{DerivedCat}_{\mathrm{bounded,coherent,Nilpotent}}(\mathrm{Stack}_{2,L,G^{\mathrm{dual}},A}/G^{\mathrm{dual}}) \tag{4.52}$$

$$\rightarrow \mathrm{DerivedCat}_{\mathrm{bounded,coherent,Nilpotent}}(\mathrm{Stack}_{L,G^{\mathrm{dual}},A}/G^{\mathrm{dual}}). \tag{4.53}$$

Remark 14. The automorphic to Galois/Weil direction of the Langlands correspondence in [LPFS] can be realized in our setting as well, but we need to consider a corresponding larger category to similarize the corresponding $\mathrm{Weil}_{K',2}$-G-equivariant sheaves coming from the smooth representations of G. For instance over $\mathrm{Bundle}_{2,G}(\mathrm{FF}_{2,\square})$ we consider such category of $\mathrm{Weil}_{K',2}$-G-equivariant sheaves not coming from the smooth representations of G, which we denote by $\mathrm{Perv}_{K',G,\mathrm{smooth},2}$, exactly as in the corresponding usual situation in [LPFS]. Then we take the corresponding $\mathrm{Weil}_{K',2}$-equivariant sheaves and cohomologies obtained from this. Then all the corresponding stacks of shtukas and the cohomologies can be defined by using the pull back along:

$$\mathrm{FF}_{2,\square} \rightarrow \mathrm{FF}_{\square} \tag{4.54}$$

after which one can extract the corresponding representations of product of $\mathrm{Weil}_{K',2}$. Then the corresponding Drinfeld's lemma in our mixed parity setting holds true as well since we just take the original Drinfeld's lemma for K' over the corresponding morphism:

$$\mathrm{Gal}_{\mathbb{Q}_p/K,2} \rightarrow \mathrm{Gal}_{\mathbb{Q}_p/K}. \tag{4.55}$$

This will gives us the corresponding mixed-parity L-parameter from $\mathrm{Weil}_{K',2}$ into $\widehat{G}(A)$. This is why we call our current framework *generalized Langlands correspondence*. At least under the well-defined geometrization, both sides of the generalized correspondence should behave quite similar to those in [LPFS].

Acknowledgements

The author thanks Professor Kedlaya for all those suggestions around the corresponding mixed-parity theoreticalization of the work of Kedlaya, Kedlaya-Liu, Kedlaya-Pottharst-Xiao. The author thanks Professor Sorensen for suggestions on the mixed-parity representation theoretic perspectives.

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