

Topologization and Functional Analytification II:
 ∞ -Categorical Motivic Constructions for Homotopical
Contexts

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Abstract

This is our second scope of the consideration on the corresponding topologization and the corresponding functional analytification. We will focus on the corresponding functorial and motivic constructions in our current consideration. We consider topological motivic derived I -adic and derived (p, I) -adic cohomologies through derived de Rham complexes of Bhatt, Guo, Illusie, Morrow, Scholze, Frobenius sheaves over Robba rings of Kedlaya-Liu in certain derived I -adic and derived (p, I) -adic geometric context as what we defined for Bambozzi-Ben-Bassat-Kremnizer ∞ -prestacks in our previous work in this series. The foundation we will work on will be based on the work of Bambozzi-Ben-Bassat-Kremnizer, Ben-Bassat-Mukherjee, Clausen-Scholze and Kelly-Kremnizer-Mukherjee, in order to promote the construction to even more general homotopical and ∞ -categorical contexts. This gives us the chance to construct the functional analytic derived prismatic cohomology and derived preperfectoidizations, as well as the functional analytic derived logarithmic prismatic cohomology and derived logarithmic preperfectoidizations after Bhatt-Scholze and Koshikawa, in the framework of Bambozzi-Ben-Bassat-Kremnizer, Ben-Bassat-Mukherjee, Clausen-Scholze and Kelly-Kremnizer-Mukherjee.

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Chapter 1

Introduction

1.1 Introduction

In our previous paper [XT1] on the intrinsic morphisms of the corresponding topological, bornological and ind-Fréchet rings, we discussed some generalization of the corresponding contexts of [Huber1] and [Huber2], along two directions. One is along some answer to a question of Kedlaya in [Ked1, Appendix 5] namely the corresponding affinoid morphisms, and the other is along some stacky understanding of these such as in [Dr1], [Dr2], [R].

Under some certain foundation, [Pau1] actually studied many parallel considerations of our project. The parallel considerations are the corresponding ind-Banach analytic spaces, the corresponding étale and pro-étale sites of ind-Banach analytic spaces, the corresponding de Rham stacks of such analytic spaces, global de Rham cohomology and comparison over ind-Banach spaces, and algebraic derived de Rham complex over the corresponding ind-Banach spaces. And the corresponding descent over the corresponding derived ind-Banach spaces. We will consider the corresponding topologization and functional analytification of the cohomology theories (topological derived de Rham after [B1], [Bei], [GL], [III1], [III2] topological logarithmic derived de Rham after Gabber such as in [B1] and [O], certainly functional analytification construction after [KL1] and [KL2] and etc). We would like to mention that our project is also largely inspired by the corresponding development in [CS1], [CS2], [FS], [Sch2] where the corresponding v -stacky consideration is extensively developed. Note that v -stacks are very significant analytic spaces, especially in the corresponding geometrization of local Langlands correspondence in [FS], where complicated derived categories are constructed through deep foundations in [Sch2] and [CS1], [CS2] on condensed sets. One certainly believes that the corresponding foundations by using normed sets in [BBBK], [BBK], [BBM], [BK] and [KKM] will also have potential applications to certainly motivic and functorial constructions in analytic geometry which are parallel to those given in [M] by using the foundation from [CS1], [CS2]. Note that the programs in [B1], [BMS] and [BS1] have already indicated that working with the corresponding correct simplicial spaces in the analytic setting is not only a pure generalization but also a very significant point around even the non simplicial analytic spaces especially in the corresponding singular situations.

Then we follow [An1], [An2], [B1], [Bei], [BS1], [G1], [GL], [III1], [III2], [Qui] to extend the corresponding discussion to certainly spaces, we mainly have focused on the corresponding

rigid analytic spaces, pseudorigid spaces in some explicit way. And we then extend the corresponding discussion to the corresponding derived setting. The corresponding derived setting is literally following [An1], [An2], [B1], [B2], [Bei], [BMS], [BS1], [G1], [GL], [III1], [III2], [Qui] and the corresponding derived logarithmic setting is literally following [B1], [Ko1] and [O]¹.

We promote the construction from Bhatt, Illusie, Guo, Morrow, Scholze, Gabber [B1], [G1], [III1], [III2], [BMS] [O] in the context of topological derived de Rham complexes and topological derived logarithmic de Rham complexes, the construction from Nicolaus-Scholze [NS] in the context of derived THH, TP and TC on the level of \mathbb{E}_1 -rings, the construction from Kedlaya-Liu in the context of the derived Robba rings and derived Frobenius sheaves [KL1] and [KL2], the construction from Bhatt-Scholze, Koshikawa and in the context of derived prismatic cohomology and derived logarithmic prismatic cohomology [BS1] and [Ko1] to the level of ∞ -categorical functional analytic level after Lurie [Lu1], [Lu2] in the stable ∞ -category of derived J -complete simplicial commutative algebras, Bambozzi-Ben-Bassat-Kremnizer [BBBK], Ben-Bassat-Mukherjee [BBM], Bambozzi-Kremnizer [BK], Clausen-Scholze [CS1] [CS2] and Kelly-Kremnizer-Mukherjee [KKM] in the stable ∞ -category of simplicial functional analytic commutative algebras.

Furthermore we promote the construction from Bhatt, Illusie, Guo, Morrow, Scholze, Gabber [B1], [G1], [III1], [III2], [BMS2] [O] in the context of topological derived de Rham complexes and topological derived logarithmic de Rham complexes, the construction from Nicolaus-Scholze [NS] in the context of derived THH, TP and TC on the level of \mathbb{E}_1 -rings, the construction from Kedlaya-Liu in the context of the derived Robba rings and derived Frobenius sheaves [KL1] and [KL2], the construction from Bhatt-Scholze, Koshikawa and in the context of derived prismatic cohomology and derived logarithmic prismatic cohomology [BS1] and [Ko1] not only to the level of ∞ -categorical functional analytic level after [Lu1], [Lu2], but also to the corresponding $(\infty, 1)$ -ringed toposes level after Lurie [Lu1], [Lu2] in the ∞ -category of ∞ -ringed toposes, Bambozzi-Ben-Bassat-Kremnizer [BBBK], Ben-Bassat-Mukherjee [BBM], Bambozzi-Kremnizer [BK], Clausen-Scholze [CS1] [CS2] and Kelly-Kremnizer-Mukherjee [KKM] in the stable ∞ -category of ∞ -functional analytic ringed toposes.

We then work in the noncommutative setting after [Kon1], [Ta], [KR1] and [KR2], with some philosophy rooted in some noncommutative motives and the corresponding nonabelian applications in noncommutative analytic geometry in the derived sense, and the noncommutative analogues of the corresponding Riemann hypothesis and the corresponding Tamagawa number conjectures, and so on. The issue is certainly that the usual Frobenius map looks somehow strange, which reminds us of the fact that actually we need to consider really large objects such as the corresponding Topological Hochschild Homologies and the corresponding nearby objects, in order to define the corresponding analogues of the corresponding prismatic cohomology through THH and nearby objects, the corresponding noncommutative 'perfectoidizations'. Here we choose to consider [NS] in order to apply the constructions to certain ∞ -rings, which we will call them Fukaya-Kato analytifications from [FK] as a noncommutative analog of the constructions in [BBM].

We first consider the following list of motivic constructions in this paper which we hope to

¹The corresponding [Ko1] has already mentioned the corresponding derived logarithmic prismatic cohomology.

establish in very general topological and Banach setting, where one can actually believe that the constructions are somehow parallel and in some sense equivalent even on the ∞ -categorical level²:

Setting 1.1.1.

- A. Derived Topological de Rham complexes and Derived Topological Logarithmic de Rham complexes of simplicial derived I -complete rings and over pro-étale site, after [B1], [B2], [Bei], [BMS], [BS1], [G1], [GL], [III1], [III2], [O];
- B. $\mathcal{O}_{\mathrm{dR}}$ -sheaves and $\mathcal{O}_{\mathrm{dR},\log}$ -sheaves over general analytic adic spaces³ after [DLLZ2], [Sch2];
- C. φ - \tilde{C}_X -sheaves, relative B -pairs over general analytic adic spaces after [KL1] and [KL2];
- D. Derived Prismatic Cohomology and Derived Logarithmic Prismatic Cohomology, after [BS1], [Ko1].

Remark 1.1.2. One could also consider the corresponding derived I -complete version⁴ of the corresponding left Kan extended THH and HH \mathbb{E}_1 -ring spectra of derived I -topological \mathbb{E}_1 -ring spectra as in [NS] and [KKM]. In certain situation, this should be able to be compared to the constructions above.

²We are actually talking about the spaces which might not be smooth.

³These are not necessarily p -adic Tate. But as long as p is topological nilpotent we do have a basis consisting of perfectoid subdomains as in [Ked1, Theorem 2.9.9, Remark 2.9.10].

⁴In fact we are taking derived I -completion of the spectra instead of the I -completion of the spectra in ∞ -category as in [Lu2, Chapter 7.3].

1.2 Preliminary

1.2.1 The Rings in Algebraic Topology

We consider some very general topological rings. We will consider the corresponding Huber's adic rings, namely we allow the corresponding rings to contain some adic open subrings which have then the corresponding linear topology. And we consider more general simplicial topological rings which are more general adic in some derived sense.

Setting 1.2.1. We consider the corresponding category $C_{\infty, E_{\infty}}$ of all the \mathbb{E}_{∞} rings⁵. We then consider the corresponding category $C_{\infty, E_{\infty}, \text{rat}, I}$ of all the \mathbb{E}_{∞} rings which have subrings being derived I -adically complete⁶, where I is finite generating set coming from elements in π_0 . And we have the smaller category $C_{\infty, E_{\infty}, \text{int}, I}$ of all the all the \mathbb{E}_{∞} rings being derived I -adically complete.

Example 1.2.2. We have many interesting ring spectra from classical algebraic topology. One can further takes the corresponding derived p -completion to achieve many interesting objects⁷. For more discussion see [MP, Chapter 5-13] and [N].

Example 1.2.3. As also discussed in [MP, Chapter 14-19] on the level of just model categories or more general construction in [Lu1, Chapter 1.3] one considers the stable homotopy category $D(R)$ of some ring spectra, namely the corresponding ∞ -enhancement of the corresponding classical triangulated categories. One considers the corresponding derived complete objects in this ∞ -category, then one will have some interesting \mathbb{E}_{∞} -spectra.

Example 1.2.4. We then have the example that is the corresponding rigid analytic affinoids in [Ta, Definition 4.1]. They have open subrings which are actually derived p -adically complete.

Example 1.2.5. We then have the example that is the corresponding pseudorigid analytic affinoids as in [Bel1, Definition 3.1], [Bel2] and [L, Definition 4.1]. They have open subrings which are actually derived t -adically complete for some specific pseudouniformizer t .

Example 1.2.6. We now fix a bounded morphism of simplicial adic rings $A \rightarrow B$ over A^* where A^* contains a corresponding ring of definition A_0^* which is complete with respect to the (p, I) -topology and we assume that A is adic and we assume that (A_0^*, I) is a prism in [BS1] namely we at least require that the corresponding δ -structure on the corresponding ring will induce the map $\varphi(\cdot) := \cdot^p + p\delta(\cdot)$ such that we have the situation where $p \in (I, \varphi(I))$. For A or B respectively we assume this contains a subring A_0 or B_0 (over A_0^*) respectively such that we have A_0 or B_0 respectively is derived complete with respect to the corresponding derived (p, I) -topology and we assume that $B = B_0[1/f, f \in I]$ (same for A). All the adic rings are assumed to be open mapping. We use the notation d to denote a corresponding primitive element as in [BS1, Section 2.3] for A^* .

⁵These are the corresponding ring objects in the algebraic topology, see [Lu1, Chapter 7]. So we want to consider the corresponding interesting objects in classical algebraic topology such as in [MP] and [N].

⁶In fact we are taking derived I -completion of the spectra instead of the I -completion of the spectra in ∞ -category as in [Lu2, Chapter 7.3].

⁷In fact we are taking derived I -completion of the spectra instead of the I -completion of the spectra in ∞ -category as in [Lu2, Chapter 7.3]. This should be different in more general setting from the consideration in algebraic topology and more general ∞ -category theory.

We also consider the corresponding simplicial prelog rings as in [B1, Chapter 6], and also consider the corresponding application of such algebraic derived constructions to some derived I -adic rings carrying the corresponding prelog structures such as in [DLLZ1, Chapter 2].

Setting 1.2.7. We consider the corresponding category $C_{\infty, E_{\infty}, \text{prelog}}$ of all the \mathbb{E}_{∞} rings carrying the corresponding prelog structures. We then consider the corresponding category $C_{\infty, E_{\infty}, \text{rat}, I, \text{prelog}}$ of all the \mathbb{E}_{∞} rings carrying prelog structures which have subrings being derived I -adically complete⁸, where I is finite generating set coming from elements in π_0 . And we have the smaller category $C_{\infty, E_{\infty}, \text{int}, I, \text{prelog}}$ of all the all the \mathbb{E}_{∞} rings carrying prelog structures being derived I -adically complete.

Example 1.2.8. The first example is the corresponding rigid analytic affinoids in [Ta, Definition 4.1]. They have open subrings which are actually derived p -adically complete. Then one adds the corresponding prelog structures from [DLLZ1, Chapter 2].

Example 1.2.9. The second example is the corresponding pseudorigid analytic affinoids as in [Bel1, Definition 3.1], [Bel2] and [L, Definition 4.1]. They have open subrings which are actually derived t -adically complete for some specific pseudouniformizer t . Then one adds the corresponding prelog structures from [DLLZ1, Chapter 2].

Example 1.2.10. We now fix a bounded morphism of logarithmic simplicial adic rings $(A, M) \rightarrow (B, N)$ over A^* where A^* contains a corresponding ring of definition A_0^* which is complete with respect to the (p, I) -topology and we assume that A is adic and we assume that (A_0^*, I) is a prism namely we at least require that the corresponding δ -structure on the corresponding ring will induce the map $\varphi(\cdot) := \cdot^p + p\delta(\cdot)$ such that we have the situation where $p \in (I, \varphi(I))$. For (A, M) or (B, N_0) respectively we assume this contains a subring (A_0, M_0) or (B_0, N_0) (over A_0^*) respectively such that we have (A_0, M_0) or (B_0, N_0) respectively is derived complete with respect to the corresponding derived (p, I) -topology and we assume that $B = B_0[1/f, f \in I]$ (same for A). All the adic rings are assumed to be open mapping. We use the notation d to denote a corresponding primitive element as in [BS1, Section 2.3] for A^* .

⁸In fact again we are taking derived I -completion of the spectra instead of the I -completion of the spectra in ∞ -category as in [Lu2, Chapter 7.3].

1.3 Notations on ∞ -Categories of ∞ -Rings

Notation	Description
$C_{\infty, \mathbb{E}_{\infty}}$	$(\infty, 1)$ -category of \mathbb{E}_{∞} rings.
$C_{\infty, \mathbb{E}_{\infty}, \text{rat}, I}$	$(\infty, 1)$ -category of \mathbb{E}_{∞} rings in the sense that the rings contain some derived rings rationally.
$C_{\infty, \mathbb{E}_{\infty}, \text{int}, I}$	$(\infty, 1)$ -category of \mathbb{E}_{∞} rings which are derived adic rings.
$C_{\infty, \mathbb{E}_{\infty}, \text{prelog}}$	$(\infty, 1)$ -category of \mathbb{E}_{∞} logarithmic rings.
$C_{\infty, \mathbb{E}_{\infty}, \text{rat}, I, \text{prelog}}$	$(\infty, 1)$ -category of \mathbb{E}_{∞} logarithmic rings in the sense that the rings contain some derived rings rationally.
$C_{\infty, \mathbb{E}_{\infty}, \text{int}, I, \text{prelog}}$	$(\infty, 1)$ -category of \mathbb{E}_{∞} logarithmic rings which are derived adic rings.
$\text{Object}_{\mathbb{E}_{\infty} \text{ commutative algebra, Simplicial}}(\text{IndSNorm}_R)$	$(\infty, 1)$ -category of \mathbb{E}_{∞} commutative algebra objects in the $(\infty, 1)$ -category of \mathbb{E}_{∞} ind-seminormed modules over R .
$\text{Object}_{\mathbb{E}_{\infty} \text{ commutative algebra, Simplicial}}(\text{Ind}^m \text{SNorm}_R)$	$(\infty, 1)$ -category of \mathbb{E}_{∞} commutative algebra objects in the $(\infty, 1)$ -category of \mathbb{E}_{∞} monomorphic ind-seminormed modules over R .
$\text{Object}_{\mathbb{E}_{\infty} \text{ commutative algebra, Simplicial}}(\text{IndNorm}_R)$	$(\infty, 1)$ -category of \mathbb{E}_{∞} commutative algebra objects in the $(\infty, 1)$ -category of \mathbb{E}_{∞} ind-normed modules over R .
$\text{Object}_{\mathbb{E}_{\infty} \text{ commutative algebra, Simplicial}}(\text{Ind}^m \text{Norm}_R)$	$(\infty, 1)$ -category of \mathbb{E}_{∞} commutative algebra objects in the $(\infty, 1)$ -category of \mathbb{E}_{∞} monomorphic ind-normed modules over R .
$\text{Object}_{\mathbb{E}_{\infty} \text{ commutative algebra, Simplicial}}(\text{IndBan}_R)$	$(\infty, 1)$ -category of \mathbb{E}_{∞} commutative algebra objects in the $(\infty, 1)$ -category of \mathbb{E}_{∞} ind-Banach modules over R .
$\text{Object}_{\mathbb{E}_{\infty} \text{ commutative algebra, Simplicial}}(\text{Ind}^m \text{Ban}_R)$	$(\infty, 1)$ -category of \mathbb{E}_{∞} commutative algebra objects in the $(\infty, 1)$ -category of \mathbb{E}_{∞} monomorphic ind-Banach modules over R .
$\text{Object}_{\mathbb{E}_{\infty} \text{ commutative algebra, Simplicial}}(\text{IndSNorm}_{\mathbb{F}_1})$	$(\infty, 1)$ -category of \mathbb{E}_{∞} commutative algebra objects in the $(\infty, 1)$ -category of \mathbb{E}_{∞} ind-seminormed sets over \mathbb{F}_1 .

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{Ind}^m\text{SNorm}_{\mathbb{F}_1})$	$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ monomorphic ind-seminormed sets over \mathbb{F}_1 .
$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{IndNorm}_{\mathbb{F}_1})$	$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ ind-normed sets over \mathbb{F}_1 .
$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{Ind}^m\text{Norm}_{\mathbb{F}_1})$	$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ monomorphic ind-normed sets over \mathbb{F}_1 .
$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{IndBan}_{\mathbb{F}_1})$	$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ ind-Banach sets over \mathbb{F}_1 .
$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{Ind}^m\text{Ban}_{\mathbb{F}_1})$	$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ monomorphic ind-Banach sets over \mathbb{F}_1 .
$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{IndSNorm}_R)^{\text{prelog}}$	$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ ind-seminormed modules over R carrying the corresponding logarithmic structures.
$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{Ind}^m\text{SNorm}_R)^{\text{prelog}}$	$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ monomorphic ind-seminormed modules over R carrying the corresponding logarithmic structures.
$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{IndNorm}_R)^{\text{prelog}}$	$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ ind-normed modules over R carrying the corresponding logarithmic structures.
$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{Ind}^m\text{Norm}_R)^{\text{prelog}}$	$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ monomorphic ind-normed modules over R carrying the corresponding logarithmic structures.

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{IndBan}_R)^{\text{prelog}}$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ ind-Banach modules over R carrying the corresponding logarithmic structures.

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{Ind}^m\text{Ban}_R)^{\text{prelog}}$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ monomorphic ind-Banach modules over R carrying the corresponding logarithmic structures.

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{IndSNorm}_{\mathbb{F}_1})^{\text{prelog}}$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ ind-seminormed sets over \mathbb{F}_1 carrying the corresponding logarithmic structures.

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{Ind}^m\text{SNorm}_{\mathbb{F}_1})^{\text{prelog}}$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ monomorphic ind-seminormed sets over \mathbb{F}_1 carrying the corresponding logarithmic structures.

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{IndNorm}_{\mathbb{F}_1})^{\text{prelog}}$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ ind-normed sets over \mathbb{F}_1 carrying the corresponding logarithmic structures.

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{Ind}^m\text{Norm}_{\mathbb{F}_1})^{\text{prelog}}$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ monomorphic ind-normed sets over \mathbb{F}_1 carrying the corresponding logarithmic structures.

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{IndBan}_{\mathbb{F}_1})^{\text{prelog}}$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ ind-Banach sets over \mathbb{F}_1 carrying the corresponding logarithmic structures.

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{Ind}^m\text{Ban}_{\mathbb{F}_1})^{\text{prelog}}$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ monomorphic ind-Banach sets over \mathbb{F}_1 carrying the corresponding logarithmic structures.

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{IndSNorm}_R)^\square$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ ind-seminormed modules over R , generated by the corresponding formal series rings over R . Here \square denotes smoothformalseriesclosure.

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{Ind}^m\text{SNorm}_R)^\square$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ monomorphic ind-seminormed modules over R , generated by the corresponding formal series rings over R .

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{IndNorm}_R)^\square$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ ind-normed modules over R , generated by the corresponding formal series rings over R , generated by the corresponding formal series rings over R .

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{Ind}^m\text{Norm}_R)^\square$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ monomorphic ind-normed modules over R , generated by the corresponding formal series rings over R .

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{IndBan}_R)^\square$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ ind-Banach modules over R , generated by the corresponding formal series rings over R .

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{Ind}^m\text{Ban}_R)^\square$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ monomorphic ind-Banach modules over R , generated by the corresponding formal series rings over R .

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{IndSNorm}_{\mathbb{F}_1})^\square$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ ind-seminormed sets over \mathbb{F}_1 , generated by the corresponding formal series rings over \mathbb{F}_1 .

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{Ind}^m\text{SNorm}_{\mathbb{F}_1})^\square$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ monomorphic ind-seminormed sets over \mathbb{F}_1 , generated by the corresponding formal series rings over \mathbb{F}_1 .

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{IndNorm}_{\mathbb{F}_1})^\square$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ ind-normed sets over \mathbb{F}_1 , generated by the corresponding formal series rings over \mathbb{F}_1 .

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{Ind}^m\text{Norm}_{\mathbb{F}_1})^\square$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ monomorphic ind-normed sets over \mathbb{F}_1 , generated by the corresponding formal series rings over \mathbb{F}_1 .

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{IndBan}_{\mathbb{F}_1})^\square$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ ind-Banach sets over \mathbb{F}_1 , generated by the corresponding formal series rings over \mathbb{F}_1 .

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{Ind}^m\text{Ban}_{\mathbb{F}_1})^\square$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ monomorphic ind-Banach sets over \mathbb{F}_1 , generated by the corresponding formal series rings over \mathbb{F}_1 .

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{IndSNorm}_R)^\square,\text{prelog}$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ ind-seminormed modules over R carrying the corresponding logarithmic structures, generated by the corresponding formal series rings over R .

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{Ind}^m\text{SNorm}_R)^\square,\text{prelog}$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ monomorphic ind-seminormed modules over R carrying the corresponding logarithmic structures, generated by the corresponding formal series rings over R .

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{IndNorm}_R)^{\square,\text{prelog}}$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ ind-normed modules over R carrying the corresponding logarithmic structures, generated by the corresponding formal series rings over R .

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{Ind}^m\text{Norm}_R)^{\square,\text{prelog}}$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ monomorphic ind-normed modules over R carrying the corresponding logarithmic structures, generated by the corresponding formal series rings over R .

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{IndBan}_R)^{\square,\text{prelog}}$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ ind-Banach modules over R carrying the corresponding logarithmic structures, generated by the corresponding formal series rings over R .

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{Ind}^m\text{Ban}_R)^{\square,\text{prelog}}$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ monomorphic ind-Banach modules over R carrying the corresponding logarithmic structures, generated by the corresponding formal series rings over R .

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{IndSNorm}_{\mathbb{F}_1})^{\square,\text{prelog}}$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ ind-seminormed sets over \mathbb{F}_1 carrying the corresponding logarithmic structures, generated by the corresponding formal series rings over \mathbb{F}_1 .

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{Ind}^m\text{SNorm}_{\mathbb{F}_1})^{\square,\text{prelog}}$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ monomorphic ind-seminormed sets over \mathbb{F}_1 carrying the corresponding logarithmic structures, generated by the corresponding formal series rings over \mathbb{F}_1 .

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{IndNorm}_{\mathbb{F}_1})^{\square,\text{prelog}}$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ ind-normed sets over \mathbb{F}_1 carrying the corresponding logarithmic structures, generated by the corresponding formal series rings over \mathbb{F}_1 .

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{Ind}^m\text{Norm}_{\mathbb{F}_1})^{\square,\text{prelog}}$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ monomorphic ind-normed sets over \mathbb{F}_1 carrying the corresponding logarithmic structures, generated by the corresponding formal series rings over \mathbb{F}_1 .

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{IndBan}_{\mathbb{F}_1})^{\square,\text{prelog}}$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ ind-Banach sets over \mathbb{F}_1 carrying the corresponding logarithmic structures, generated by the corresponding formal series rings over \mathbb{F}_1 .

$\text{Object}_{\mathbb{E}_\infty\text{commutativealgebra,Simplicial}}(\text{Ind}^m\text{Ban}_{\mathbb{F}_1})^{\square,\text{prelog}}$

$(\infty,1)$ -category of \mathbb{E}_∞ commutative algebra objects in the $(\infty,1)$ -category of \mathbb{E}_∞ monomorphic ind-Banach sets over \mathbb{F}_1 carrying the corresponding logarithmic structures, generated by the corresponding formal series rings over \mathbb{F}_1 .

Remark 1.3.1. The generating process above is by adding all the colimits in the homotopy sense which are assumed to be sifted. R will be Banach and carrying the corresponding p -adic topology.

1.4 Notations on ∞ -Categories of Commutative ∞ -Ringed Toposes

We change the notations for ∞ -ringed toposes slightly, therefore let us start from the rings.

Rings:

$\text{Ind}^{\text{smoothformalseriesclosure}}\text{Commutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_R)$: $(\infty, 1)$ -category of simplicial commutative algebra objects in the ∞ -category of colimit completion of seminormed modules over a general Banach ring R , generated by the corresponding power series rings by taking colimits in the homotopical sense.

$\text{Ind}^{\text{smoothformalseriesclosure}}\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m\text{Seminormed}_R)$: $(\infty, 1)$ -category of simplicial commutative algebra objects in the ∞ -category of monomorphic colimit completion of seminormed modules over a general Banach ring R , generated by the corresponding power series rings by taking colimits in the homotopical sense.

$\text{Ind}^{\text{smoothformalseriesclosure}}\text{Commutativealgebra}_{\text{simplicial}}(\text{IndNormed}_R)$: $(\infty, 1)$ -category of simplicial commutative algebra objects in the ∞ -category of colimit completion of normed modules over a general Banach ring R , generated by the corresponding power series rings by taking colimits in the homotopical sense.

$\text{Ind}^{\text{smoothformalseriesclosure}}\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m\text{Normed}_R)$: $(\infty, 1)$ -category of simplicial commutative algebra objects in the ∞ -category of monomorphic colimit completion of normed modules over a general Banach ring R , generated by the corresponding power series rings by taking colimits in the homotopical sense.

$\text{Ind}^{\text{smoothformalseriesclosure}}\text{Commutativealgebra}_{\text{simplicial}}(\text{IndBanach}_R)$: $(\infty, 1)$ -category of simplicial commutative algebra objects in the ∞ -category of colimit completion of Banach modules over a general Banach ring R , generated by the corresponding power series rings by taking colimits in the homotopical sense.

$\text{Ind}^{\text{smoothformalseriesclosure}}\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m\text{Banach}_R)$: $(\infty, 1)$ -category of simplicial commutative algebra objects in the ∞ -category of monomorphic colimit completion of Banach modules over a general Banach ring R , generated by the corresponding power series rings by taking colimits in the homotopical sense.

$\text{Ind}^{\text{smoothformalseriesclosure}}\text{Commutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_{\mathbb{F}_1})$: $(\infty, 1)$ -category of simplicial commutative algebra objects in the ∞ -category of colimit completion of seminormed sets over a general Banach ring \mathbb{F}_1 , generated by the corresponding power series rings by taking colimits in the homotopical sense.

$\text{Ind}^{\text{smoothformalseriesclosure}}\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m\text{Seminormed}_{\mathbb{F}_1})$: $(\infty, 1)$ -category of simplicial commutative algebra objects in the ∞ -category of monomorphic colimit completion of seminormed sets over a general Banach ring \mathbb{F}_1 , generated by the corresponding power series rings by taking colimits in the homotopical sense.

$\text{Ind}^{\text{smoothformalseriesclosure}}\text{Commutativealgebra}_{\text{simplicial}}(\text{IndNormed}_{\mathbb{F}_1})$: $(\infty, 1)$ -category of simplicial commutative algebra objects in the ∞ -category of colimit completion of normed sets over a general Banach ring \mathbb{F}_1 , generated by the corresponding power series rings by taking colimits in the homotopical sense.

$\text{Ind}^{\text{smoothformalseriesclosure}}\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m\text{Normed}_{\mathbb{F}_1})$: $(\infty, 1)$ -category of simplicial commutative algebra objects in the ∞ -category of monomorphic colimit completion of

normed sets over a general Banach ring \mathbb{F}_1 , generated by the corresponding power series rings by taking colimits in the homotopical sense.

$\text{Ind}^{\text{smoothformalseriesclosure}} \text{Commutativealgebra}_{\text{simplicial}}(\text{IndBanach}_{\mathbb{F}_1})$: $(\infty, 1)$ -category of simplicial commutative algebra objects in the ∞ -category of colimit completion of Banach sets over a general Banach ring \mathbb{F}_1 , generated by the corresponding power series rings by taking colimits in the homotopical sense.

$\text{Ind}^{\text{smoothformalseriesclosure}} \text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_{\mathbb{F}_1})$: $(\infty, 1)$ -category of simplicial commutative algebra objects in the ∞ -category of monomorphic colimit completion of Banach sets over a general Banach ring \mathbb{F}_1 , generated by the corresponding power series rings by taking colimits in the homotopical sense.

Prestacks:

∞ – $\text{Prestack}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}$: ∞ -presheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$.

∞ – $\text{Prestack}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}$: ∞ -presheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$.

∞ – $\text{Prestack}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{IndNormed}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}$: ∞ -presheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$.

∞ – $\text{Prestack}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}$: ∞ -presheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$.

∞ – $\text{Prestack}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{IndBanach}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}$: ∞ -presheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$.

∞ – $\text{Prestack}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}$: ∞ -presheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$.

Stacks:

∞ – $\text{Stack}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}$: ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$.

∞ – $\text{Stack}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}$: ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$.

∞ – $\text{Stack}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{IndNormed}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}$: ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$.

∞ – $\text{Stack}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}$: ∞ -sheaves into ∞ -

groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$.

∞ – $\text{Stack}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{IndBanach}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}$: ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$.

∞ – $\text{Stack}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}$: ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$.

Ringed Toposes:

∞ – $\text{Toposes}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}^{\text{ringed}, \text{commutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_?)}$: ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure.

∞ – $\text{Toposes}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}^{\text{ringed}, \text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_?)}$: ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure.

∞ – $\text{Toposes}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{IndNormed}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}^{\text{ringed}, \text{Commutativealgebra}_{\text{simplicial}}(\text{IndNormed}_?)}$: ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure.

∞ – $\text{Toposes}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}^{\text{ringed}, \text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_?)}$: ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure.

∞ – $\text{Toposes}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{IndBanach}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}^{\text{ringed}, \text{Commutativealgebra}_{\text{simplicial}}(\text{IndBanach}_?)}$: ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure.

∞ – $\text{Toposes}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}^{\text{ringed}, \text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_?)}$: ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure.

$\text{Proj}^{\text{smoothformalseriesclosure}}$ – $\text{Toposes}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}^{\text{ringed}, \text{commutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_?)}$: ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure by specializing a ring object \mathcal{O} .

$\text{Proj}^{\text{smoothformalseriesclosure}}_{\infty} - \text{Toposes}_{\text{ringed,Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_?)}$
 $\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_?)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}$
 ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure by specializing a ring object \mathcal{O} .

$\text{Proj}^{\text{smoothformalseriesclosure}}_{\infty} - \text{Toposes}_{\text{ringed,Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_?)}$
 $\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_?)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}$
 ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure by specializing a ring object \mathcal{O} .

$\text{Proj}^{\text{smoothformalseriesclosure}}_{\infty} - \text{Toposes}_{\text{ringed,Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_?)}$
 $\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_?)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}$
 ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure by specializing a ring object \mathcal{O} .

$\text{Proj}^{\text{smoothformalseriesclosure}}_{\infty} - \text{Toposes}_{\text{ringed,Commutativealgebra}_{\text{simplicial}}(\text{Ind} \text{Banach}_?)}$
 $\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind} \text{Banach}_?)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}$
 ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure by specializing a ring object \mathcal{O} .

$\text{Proj}^{\text{smoothformalseriesclosure}}_{\infty} - \text{Toposes}_{\text{ringed,Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_?)}$
 $\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_?)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}$
 ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure by specializing a ring object \mathcal{O} .

Remark 1.4.1. In p -adic Hodge theory, one usually will need to construct presheaves M out of from the ∞ -ring object \mathcal{O} . Also one can define the corresponding pre- ∞ -ringed Toposes, we will not continue provide the corresponding group of notations in the parallel way.

∞ -Quasicoherent Sheaves of Functional Analytic Modules over Ringed Toposes $\sharp = \text{Seminormed, Normed, Banach}$

$\text{Ind}\sharp\text{Quasicoherent}_{\infty} - \text{Toposes}_{\text{ringed,commutativealgebra}_{\text{simplicial}}(\text{Ind}\text{Seminormed}_?)}$: Colimits completion of ∞ -Quasicoherent Sheaves of Functional Analytic Modules over ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure.

$\text{Ind}\sharp\text{Quasicoherent}_{\infty} - \text{Toposes}_{\text{ringed,Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_?)}$: Colimits completion of ∞ -Quasicoherent Sheaves of Functional Analytic Modules over ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure.

$\text{Ind}\sharp\text{Quasicoherent}_{\infty} - \text{Toposes}_{\text{ringed,Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_?)}$: Colimits completion of ∞ -Quasicoherent Sheaves of Functional Analytic Modules over ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure.

$\text{Ind}\sharp\text{Quasicoherent}_{\infty} - \text{Toposes}_{\text{ringed,Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_?)}$: Colimits completion of ∞ -Quasicoherent Sheaves of Functional Analytic Modules over ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure.

$\text{Ind}\#_{\text{Toposes}}^{\text{ringed,Commutativealgebra}_{\text{simplicial}}(\text{IndNormed}_?)}$: Colimits completion of ∞ -Quasicoherent Sheaves of Functional Analytic Modules over ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure.

$\text{Ind}\#_{\text{Toposes}}^{\text{ringed,Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m\text{Normed}_?)}$: Colimits completion of ∞ -Quasicoherent Sheaves of Functional Analytic Modules over ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure.

$\text{Ind}\#_{\text{Toposes}}^{\text{ringed,Commutativealgebra}_{\text{simplicial}}(\text{IndBanach}_?)}$: Colimits completion of ∞ -Quasicoherent Sheaves of Functional Analytic Modules over ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure.

$\text{Ind}\#_{\text{Toposes}}^{\text{ringed,Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m\text{Banach}_?)}$: Colimits completion of ∞ -Quasicoherent Sheaves of Functional Analytic Modules over ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure.

$\text{Ind}\#_{\text{Ind}^{\text{smoothformalseriesclosure}}\infty\text{-Toposes}}^{\text{ringed,commutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_?)}$: Colimits completion of ∞ -Quasicoherent Sheaves of Functional Analytic Modules over ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure by specializing a ring object \mathcal{O} . The difference is that we consider the corresponding \mathcal{O} living in the colimit completion closure.

$\text{Ind}\#_{\text{Ind}^{\text{smoothformalseriesclosure}}\infty\text{-Toposes}}^{\text{ringed,Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m\text{Seminormed}_?)}$: Colimits completion of ∞ -Quasicoherent Sheaves of Functional Analytic Modules over ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure by specializing a ring object \mathcal{O} . The difference is that we consider the corresponding \mathcal{O} living in the colimit completion closure.

$\text{Ind}\#_{\text{Ind}^{\text{smoothformalseriesclosure}}\infty\text{-Toposes}}^{\text{ringed,Commutativealgebra}_{\text{simplicial}}(\text{IndNormed}_?)}$: Colimits completion of ∞ -Quasicoherent Sheaves of Functional Analytic Modules over ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure by specializing

a ring object \mathcal{O} . The difference is that we consider the corresponding \mathcal{O} living in the colimit completion closure.

$\text{Ind}\sharp\text{Quasicoherent}_{\text{Ind}^{\text{smoothformalseriesclosure}}\infty\text{-Toposes}}^{\text{ringed,Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m\text{Normed?})} : \text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m\text{Normed?})^{\text{opposite,Grotopology,homotopyepimorphism}}$

Colimits completion of ∞ -Quasicoherent Sheaves of Functional Analytic Modules over ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure by specializing a ring object \mathcal{O} . The difference is that we consider the corresponding \mathcal{O} living in the colimit completion closure.

$\text{Ind}\sharp\text{Quasicoherent}_{\text{Ind}^{\text{smoothformalseriesclosure}}\infty\text{-Toposes}}^{\text{ringed,Commutativealgebra}_{\text{simplicial}}(\text{IndBanach?})} : \text{Commutativealgebra}_{\text{simplicial}}(\text{IndBanach?})^{\text{opposite,Grotopology,homotopyepimorphism}}$

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1.5 Notations on ∞ -Categories of Noncommutative ∞ -Ringed Toposes

We change the notations for ∞ -ringed noncommutative toposes slightly, therefore let us start from the rings. The corresponding generators we will choose in order to take the corresponding homotopy colimit completion and the corresponding homotopy limit completion are Fukaya-Kato rings in [FK]:

$$R \langle Z_1, \dots, Z_n \rangle, n \geq 1, \quad (1.5.1)$$

$$R [[Z_1, \dots, Z_n]], n \geq 1, \quad (1.5.2)$$

with Z_1, \dots, Z_n are noncommuting free variables. This would be the specific completions of the polynomials:

$$R [Z_1, \dots, Z_n], n \geq 1. \quad (1.5.3)$$

$$(1.5.4)$$

The corresponding analogs of analytification functors from Ben-Bassat-Mukherjee [BBM, Section 4.2] are given in the following. First we consider the ∞ -category of \mathbb{E}_1 -rings from [Lu2, Proposition 7.1.4.18, as well as the discussion above Proposition 7.1.4.18 on page 1225] which we denote it by $\text{Noncommutative}_{\mathbb{E}_1, \text{Simplicial}}$, then we consider the corresponding category of all the polynomial rings with free variables over R , which we denote it by $\text{Polynomial}_R^{\text{free}}$, then we have the corresponding fully faithful embedding:

$$\text{Polynomial}_R^{\text{free}} \rightarrow \text{Noncommutative}_{\mathbb{E}_1, \text{Simplicial}}. \quad (1.5.5)$$

Then take the corresponding completion for each:

$$R [Z_1, \dots, Z_n], n \geq 1, \quad (1.5.6)$$

$$(1.5.7)$$

we have the Fukaya-Kato adic ring:

$$R \langle Z_1, \dots, Z_n \rangle, n \geq 1, \quad (1.5.8)$$

$$R [[Z_1, \dots, Z_n]], n \geq 1. \quad (1.5.9)$$

As in [BBM, Section 4.2], this will give the process what we call smooth formal series analytification by taking into account the corresponding homotopy colimit completion:

$$\text{Ind}^{\text{smoothformalseriesclosure}} \text{Polynomial}_R^{\text{free}} \rightarrow \text{Noncommutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_R), \quad (1.5.10)$$

$$\text{Ind}^{\text{smoothformalseriesclosure}} \text{Polynomial}_R^{\text{free}} \rightarrow \text{Noncommutativealgebra}_{\text{simplicial}}(\text{IndNormed}_R), \quad (1.5.11)$$

$$\text{Ind}^{\text{smoothformalseriesclosure}} \text{Polynomial}_R^{\text{free}} \rightarrow \text{Noncommutativealgebra}_{\text{simplicial}}(\text{IndBanach}_R). \quad (1.5.12)$$

$$(1.5.13)$$

Remark 1.5.1. The left actually spans all the \mathbb{E}_1 -algebra in our setting by regarding the free variable polynomials as tensor algebras over R as explained in [Lu1, Proposition 7.1.4.18, as well as the discussion above Proposition 7.1.4.18 on page 1225] in analogy of [Lu1, Proposition 7.1.4.20, as well as the discussion above Proposition 7.1.4.20] in the commutative situation.

Noncommutative Rings:

$\text{Ind}^{\text{smoothformalseriesclosure}}\text{Noncommutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_R)$: $(\infty, 1)$ -category of simplicial noncommutative algebra objects in the ∞ -category of colimit completion of seminormed modules over a general Banach ring R , generated by the corresponding power series rings by taking colimits in the homotopical sense.

$\text{Ind}^{\text{smoothformalseriesclosure}}\text{Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m\text{Seminormed}_R)$: $(\infty, 1)$ -category of simplicial noncommutative algebra objects in the ∞ -category of monomorphic colimit completion of seminormed modules over a general Banach ring R , generated by the corresponding power series rings by taking colimits in the homotopical sense.

$\text{Ind}^{\text{smoothformalseriesclosure}}\text{Noncommutativealgebra}_{\text{simplicial}}(\text{IndNormed}_R)$: $(\infty, 1)$ -category of simplicial noncommutative algebra objects in the ∞ -category of colimit completion of normed modules over a general Banach ring R , generated by the corresponding power series rings by taking colimits in the homotopical sense.

$\text{Ind}^{\text{smoothformalseriesclosure}}\text{Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m\text{Normed}_R)$: $(\infty, 1)$ -category of simplicial noncommutative algebra objects in the ∞ -category of monomorphic colimit completion of normed modules over a general Banach ring R , generated by the corresponding power series rings by taking colimits in the homotopical sense.

$\text{Ind}^{\text{smoothformalseriesclosure}}\text{Noncommutativealgebra}_{\text{simplicial}}(\text{IndBanach}_R)$: $(\infty, 1)$ -category of simplicial noncommutative algebra objects in the ∞ -category of colimit completion of Banach modules over a general Banach ring R , generated by the corresponding power series rings by taking colimits in the homotopical sense.

$\text{Ind}^{\text{smoothformalseriesclosure}}\text{Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m\text{Banach}_R)$: $(\infty, 1)$ -category of simplicial noncommutative algebra objects in the ∞ -category of monomorphic colimit completion of Banach modules over a general Banach ring R , generated by the corresponding power series rings by taking colimits in the homotopical sense.

$\text{Ind}^{\text{smoothformalseriesclosure}}\text{Noncommutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_{\mathbb{F}_1})$: $(\infty, 1)$ -category of simplicial noncommutative algebra objects in the ∞ -category of colimit completion of seminormed sets over a general Banach ring \mathbb{F}_1 , generated by the corresponding power series rings by taking colimits in the homotopical sense.

$\text{Ind}^{\text{smoothformalseriesclosure}}\text{Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m\text{Seminormed}_{\mathbb{F}_1})$: $(\infty, 1)$ -category of simplicial noncommutative algebra objects in the ∞ -category of monomorphic colimit completion of seminormed sets over a general Banach ring \mathbb{F}_1 , generated by the corresponding power series rings by taking colimits in the homotopical sense.

$\text{Ind}^{\text{smoothformalseriesclosure}}\text{Noncommutativealgebra}_{\text{simplicial}}(\text{IndNormed}_{\mathbb{F}_1})$: $(\infty, 1)$ -category of simplicial noncommutative algebra objects in the ∞ -category of colimit completion of normed sets over a general Banach ring \mathbb{F}_1 , generated by the corresponding power series rings by taking colimits in the homotopical sense.

$\text{Ind}^{\text{smoothformalseriesclosure}}\text{Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m\text{Normed}_{\mathbb{F}_1})$: $(\infty, 1)$ -category of simplicial noncommutative algebra objects in the ∞ -category of monomorphic colimit completion of normed sets over a general Banach ring \mathbb{F}_1 , generated by the corresponding power series rings by taking colimits in the homotopical sense.

$\text{Ind}^{\text{smoothformalseriesclosure}}\text{Noncommutativealgebra}_{\text{simplicial}}(\text{IndBanach}_{\mathbb{F}_1})$: $(\infty, 1)$ -category of simplicial noncommutative algebra objects in the ∞ -category of colimit completion of Banach sets

over a general Banach ring \mathbb{F}_1 , generated by the corresponding power series rings by taking colimits in the homotopical sense.

$\text{Ind}^{\text{smoothformalseriesclosure}} \text{Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_{\mathbb{F}_1})$: $(\infty, 1)$ -category of simplicial noncommutative algebra objects in the ∞ -category of monomorphic colimit completion of Banach sets over a general Banach ring \mathbb{F}_1 , generated by the corresponding power series rings by taking colimits in the homotopical sense.

Prestacks:

∞ – $\text{Prestack}_{\text{Noncommutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}$: ∞ -presheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$.

∞ – $\text{Prestack}_{\text{Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}$: ∞ -presheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$.

∞ – $\text{Prestack}_{\text{Noncommutativealgebra}_{\text{simplicial}}(\text{IndNormed}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}$: ∞ -presheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$.

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∞ – $\text{Prestack}_{\text{Noncommutativealgebra}_{\text{simplicial}}(\text{IndBanach}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}$: ∞ -presheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$.

∞ – $\text{Prestack}_{\text{Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}$: ∞ -presheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$.

Stacks:

∞ – $\text{Stack}_{\text{Noncommutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}$: ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$.

∞ – $\text{Stack}_{\text{Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}$: ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$.

∞ – $\text{Stack}_{\text{Noncommutativealgebra}_{\text{simplicial}}(\text{IndNormed}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}$: ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$.

∞ – $\text{Stack}_{\text{Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_?)^{\text{opposite}}, \text{Grotopology}, \text{homotopyepimorphism}}$: ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$.

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topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$.
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The following is a noncommutative analog of [BBBK, Definition 5.6]:

Definition 1.5.2. Here a homotopy epimorphism is defined to be such a morphism $A \rightarrow B$ such that $B \otimes_A B^{\text{opp}} \rightarrow B$ reflects isomorphism in the homotopy category.

Ringed Toposes:

$\infty - \text{Toposes}_{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_?)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}}$: ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure.

$\infty - \text{Toposes}_{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_?)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}}$: ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure.

$\infty - \text{Toposes}_{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{IndNormed}_?)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}}$: ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure.

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$\infty - \text{Toposes}_{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_?)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}}$: ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure.

$\text{Proj}^{\text{smoothformalseriesclosure}} \infty - \text{Toposes}_{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_?)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}}$: ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure by specializing a ring object \mathcal{O} .

$\text{Proj}^{\text{smoothformalseriesclosure}} \infty - \text{Toposes}_{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_?)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}}$:

we assume the stack carries ∞ -ringed toposes structure.

$\text{Ind}\#_{\infty\text{-Toposes}}^{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_?)}$: Colimits completion of ∞ -Quasicoherent Sheaves of Functional Analytic Modules over ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure.

$\text{Ind}\#_{\infty\text{-Toposes}}^{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_?)}$: Colimits completion of ∞ -Quasicoherent Sheaves of Functional Analytic Modules over ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure.

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$\text{Ind}\#_{\text{Ind}^{\text{smoothformalseriesclosure}} \infty\text{-Toposes}}^{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{Ind} \text{Seminormed}_?)}$:

Colimits completion of ∞ -Quasicoherent Sheaves of Functional Analytic Modules over ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure by specializing a ring object \mathcal{O} . The difference is that we consider the corresponding \mathcal{O} living in the colimit completion closure.

$\text{Ind}\#_{\text{Ind}^{\text{smoothformalseriesclosure}} \infty\text{-Toposes}}^{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_?)}$:

Colimits completion of ∞ -Quasicoherent Sheaves of Functional Analytic Modules over ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure by specializing a ring object \mathcal{O} . The difference is that we consider the corresponding \mathcal{O} living in the colimit completion closure.

$\text{Ind}\#_{\text{Ind}^{\text{smoothformalseriesclosure}} \infty\text{-Toposes}}^{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{Ind} \text{Normed}_?)}$:

Colimits completion of ∞ -Quasicoherent Sheaves of Functional Analytic Modules over ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure by specializing a ring object \mathcal{O} . The difference is that we consider the corresponding \mathcal{O} living in the colimit completion closure.

$\text{Ind}\#_{\text{Ind}^{\text{smoothformalseriesclosure}} \infty\text{-Toposes}}^{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_?)}$:

Colimits completion of ∞ -Quasicoherent Sheaves of Functional Analytic Modules over ∞ -

sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure by specializing a ring object \mathcal{O} . The difference is that we consider the corresponding \mathcal{O} living in the colimit completion closure.

$\text{Ind}\sharp\text{Quasicoherent}$ $\text{Ind}^{\text{smoothformalseriesclosure}}\infty\text{-Toposes}^{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{IndBanach}_?)}$ $\text{Noncommutativealgebra}_{\text{simplicial}}(\text{IndBanach}_?)^{\text{opposite, Grotology, homotopyepimorphism}}$:

Colimits completion of ∞ -Quasicoherent Sheaves of Functional Analytic Modules over ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure by specializing a ring object \mathcal{O} . The difference is that we consider the corresponding \mathcal{O} living in the colimit completion closure.

$\text{Ind}\sharp\text{Quasicoherent}$ $\text{Ind}^{\text{smoothformalseriesclosure}}\infty\text{-Toposes}^{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m\text{Banach}_?)}$ $\text{Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m\text{Banach}_?)^{\text{opposite, Grotology, homotopyepimorphism}}$:

Colimits completion of ∞ -Quasicoherent Sheaves of Functional Analytic Modules over ∞ -sheaves into ∞ -groupoid over corresponding opposite category carrying the corresponding Grothendieck topology, and we will mainly consider the corresponding homotopy epimorphisms. $? = R, \mathbb{F}_1$. And we assume the stack carries ∞ -ringed toposes structure by specializing a ring object \mathcal{O} . The difference is that we consider the corresponding \mathcal{O} living in the colimit completion closure.

Let us explain slightly what is happening here, the base ∞ -rings are noncommutative in certain sense, which is not the same as in the foundation of [BBK], [BBM], [KKM], [BK]. Certainly for instance one considers the corresponding ∞ -category $\text{Simplicial}(\text{IndBanach}_{\mathbb{F}_1})$, then takes the corresponding fibrations over the corresponding noncommutative rings to achieve so.

Chapter 2

Topological Theory

2.1 Topological André-Quillen Homology and Topological Derived de Rham Complexes

2.1.1 Derived p -Complete Derived de Rham Complex

We now first discuss the corresponding Banach version of André-Quillen Homology and the corresponding Banach version of Derived de Rham complex parallel to [An1, Chapitre 3], [An2], [B1, Chapter 2, Chapter 8], [Bei, Chapter 1], [G1, Chapter 5], [GL, Chapter 3, Chapter 4], [Ill1, Chapitre II, Chapitre III], [Ill2, Chapitre VIII], [Qui, Section 4]. We would like to start from the corresponding context of [GL, Chapter 3, Chapter 4], and represent the construction for the convenience of the readers. We start from the corresponding construction of the algebraic p -adic derived de Rham complex for a map $A \rightarrow B$ of p -complete rings. This is the corresponding derived differential complex attached to the polynomial resolution of B :

$$A[A[B]]\dots, \quad (2.1.1)$$

which is now denoted by $\text{Kan}_{\text{Left}}\text{deRham}_{B/A,\text{alg}}^{\text{degreenumber}} := \text{Kan}_{\text{Left}}\text{deRham}_{-/A,\text{alg}}^{\text{degreenumber}}(B)$ after taking the corresponding left Kan extension which will be the same for all the following constructions ¹. The corresponding cotangent complex associated is defined to be just:

$$\mathbb{L}_{B/A,\text{alg}} := \text{deRham}_{A[B]^{\text{degreenumber}}/A,\text{alg}}^1 \otimes_{A[B]^{\text{degreenumber}}} B. \quad (2.1.2)$$

The corresponding algebraic André-Quillen homologies are defined to be:

$$H_{\text{degreenumber},\text{AQ,alg}} := \pi_{\text{degreenumber}}(\mathbb{L}_{B/A,\text{alg}}). \quad (2.1.3)$$

The corresponding topological André-Quillen complex is actually the completed version of the corresponding algebraic ones above by considering the corresponding certain p -completion over the simplicial module structure.

Then we consider the corresponding derived algebraic de Rham complex which is just defined to be:

$$\text{Kan}_{\text{Left}}\text{deRham}_{B/A,\text{alg}}^{\text{degreenumber}}, \text{Kan}_{\text{Left}}\text{Fil}_{\text{deRham}_{B/A,\text{alg}}^{\text{degreenumber}}}^* \quad (2.1.4)$$

¹We have already considered the corresponding left Kan extension to all the rings which are not concentrated at degree zero after [BMS, Example 5.11, Example 5.12], which is also discussed in [B2, Lecture 7].

We then take the corresponding Banach completion and we denote that by:

$$\text{Kan}_{\text{Left}}\widehat{\text{deRham}}_{B/A,\text{topo}}^{\text{degreenumber}}, \text{Kan}_{\text{Left}}\widehat{\text{Fil}}_{\text{deRham}_{B/A,\text{topo}}^{\text{degreenumber}}}^* \quad (2.1.5)$$

Then we need to take the corresponding Hodge-Filtered completion by using the corresponding filtration associated as above:

$$\text{Kan}_{\text{Left}}\widehat{\widehat{\text{deRham}}}_{B/A,\text{topo}}^{\text{degreenumber}}, \text{Kan}_{\text{Left}}\widehat{\widehat{\text{Fil}}}_{\widehat{\text{deRham}}_{B/A,\text{topo}}^{\text{degreenumber}}}^* \quad (2.1.6)$$

This is basically the corresponding analytic and complete version the corresponding algebraic de Rham complex. Furthermore we allow large coefficients with rigid affinoid algebra Z over \mathbb{Q}_p . Therefore we take the corresponding completed tensor product in the following.

Definition 2.1.1. We define the following Z deformed version of the corresponding complete version of the corresponding André-Quillen homology and the corresponding complete version of the corresponding derived de Rham complex. We start from the corresponding construction of the algebraic p -adic derived de Rham complex for a map $A \rightarrow B$. Fix a pair of ring of definitions A_0, B_0 in A, B respectively. Then this is the corresponding derived differential complex attached to the polynomial resolution of B_0 :

$$A_0[A_0[B_0]]\dots, \quad (2.1.7)$$

which is now denoted by $\text{Kan}_{\text{Left}}\widehat{\text{deRham}}_{B_0/A_0,\text{alg}}^{\text{degreenumber}}$. The corresponding cotangent complex associated is defined to be just:

$$\mathbb{L}_{B_0/A_0,\text{alg}} := \text{deRham}_{A_0[B_0]^{\text{degreenumber}}/A_0,\text{alg}}^1 \otimes_{A_0[B_0]^{\text{degreenumber}}} B_0. \quad (2.1.8)$$

The corresponding algebraic André-Quillen homologies are defined to be:

$$H_{\text{degreenumber},\text{AQ,alg}} := \pi_{\text{degreenumber}}(\mathbb{L}_{B_0/A_0,\text{alg}}). \quad (2.1.9)$$

The corresponding topological André-Quillen complex is actually the complete version of the corresponding algebraic ones above by considering the corresponding derived p -completion over the simplicial module structure:

$$\mathbb{L}_{B_0/A_0,\text{topo}} := R\varprojlim_k \text{Kos}_{p^k} \left((\text{deRham}_{A_0[B_0]^{\text{degreenumber}}/A_0,\text{alg}}^1 \otimes_{A_0[B_0]^{\text{degreenumber}}} B_0) \right). \quad (2.1.10)$$

Taking the product with \mathcal{O}_Z we have the corresponding integral version of the topological André-Quillen complex:

$$\mathbb{L}_{B_0/A_0,\text{topo},Z} := \mathbb{L}_{B_0/A_0,\text{topo}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_Z \quad (2.1.11)$$

Then we consider the corresponding derived algebraic de Rham complex which is just defined to be:

$$\text{Kan}_{\text{Left}}\widehat{\text{deRham}}_{B_0/A_0,\text{alg}}^{\text{degreenumber}}, \text{Kan}_{\text{Left}}\widehat{\text{Fil}}_{\widehat{\text{deRham}}_{B_0/A_0,\text{alg}}^1}^* \quad (2.1.12)$$

We then take the corresponding derived p -completion and we denote that by:

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{B_0/A_0,\mathrm{topo}}^{\mathrm{degreenumber}} := R\varprojlim_k \mathrm{Kos}_{p^k} \left(\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{B_0/A_0,\mathrm{alg}}^{\mathrm{degreenumber}}, \mathrm{Kan}_{\mathrm{Left}}\mathrm{Fil}_{\mathrm{deRham}_{B_0/A_0,\mathrm{alg}}^{\mathrm{degreenumber}}}^* \right), \quad (2.1.13)$$

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{Fil}_{\mathrm{deRham}_{B_0/A_0,\mathrm{topo}}^{\mathrm{degreenumber}}}^* := R\varprojlim_k \mathrm{Kos}_{p^k} \left(\mathrm{Kan}_{\mathrm{Left}}\mathrm{Fil}_{\mathrm{deRham}_{B_0/A_0,\mathrm{alg}}^{\mathrm{degreenumber}}}^* \right). \quad (2.1.14)$$

Before considering the corresponding integral version we just consider the corresponding product of these \mathbb{E}_∞ -rings with \mathcal{O}_Z to get:

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{B_0/A_0,\mathrm{topo},Z}^{\mathrm{degreenumber}} := \mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{B_0/A_0,\mathrm{topo}}^{\mathrm{degreenumber}} \widehat{\otimes}_{\mathbb{Z}_p}^{\mathbb{L}} \mathcal{O}_Z, \quad (2.1.15)$$

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{Fil}_{\mathrm{deRham}_{B_0/A_0,\mathrm{topo},Z}^{\mathrm{degreenumber}}}^* := \mathrm{Kan}_{\mathrm{Left}}\mathrm{Fil}_{\mathrm{deRham}_{B_0/A_0,\mathrm{topo}}^{\mathrm{degreenumber}}}^* \widehat{\otimes}_{\mathbb{Z}_p}^{\mathbb{L}} \mathcal{O}_Z. \quad (2.1.16)$$

Then we consider the following construction for the map $A \rightarrow B$ by putting:

$$\mathbb{L}_{B_0/A_0,\mathrm{topo},Z} := \mathrm{Colim}_{A_0 \rightarrow B_0} \mathbb{L}_{B_0/A_0,\mathrm{topo},Z}[1/p], \quad (2.1.17)$$

$$H_{\mathrm{degreenumber},\mathrm{AQ},\mathrm{topo},Z} := \pi_{\mathrm{degreenumber}}(\mathbb{L}_{B/A,\mathrm{topo},Z}), \quad (2.1.18)$$

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{B/A,\mathrm{topo},Z}^{\mathrm{degreenumber}} := \mathrm{Colim}_{A_0 \rightarrow B_0} \mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{B_0/A_0,\mathrm{topo},Z}^{\mathrm{degreenumber}}[1/p], \quad (2.1.19)$$

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{Fil}_{\mathrm{deRham}_{B/A,\mathrm{topo},Z}^{\mathrm{degreenumber}}}^* := \mathrm{Colim}_{A_0 \rightarrow B_0} \mathrm{Kan}_{\mathrm{Left}}\mathrm{Fil}_{\mathrm{deRham}_{B_0/A_0,\mathrm{topo},Z}^{\mathrm{degreenumber}}}^*[1/p]. \quad (2.1.20)$$

Then we need to take the corresponding Hodge-Filtered completion by using the corresponding filtration associated as above to achieve the corresponding Hodge-complete objects in the corresponding filtered ∞ -categories:

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{B/A,\mathrm{topo},\mathrm{Hodge},Z}^{\mathrm{degreenumber}}, \mathrm{Kan}_{\mathrm{Left}}\mathrm{Fil}_{\mathrm{deRham}_{B/A,\mathrm{topo},\mathrm{Hodge},Z}^{\mathrm{degreenumber}}}^*. \quad (2.1.21)$$

Definition 2.1.2. We define the corresponding finite projective filtered crystals to be the corresponding finite projective module spectra over the topological filtered E_∞ -ring $\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{B/A,\mathrm{topo},Z}^{\mathrm{degreenumber}}$ with the corresponding induced filtrations.

Definition 2.1.3. We define the corresponding almost perfect² filtered crystals to be the corresponding almost perfect module spectra over the topological filtered E_∞ -ring $\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{B/A,\mathrm{topo},Z}^{\mathrm{degreenumber}}$ with the corresponding induced filtrations.

The following is derived from the main Poincaré Lemma from [GL, Theorem 1.2] in the non-deformed situation. Consider a corresponding smooth rigid analytic space X over k/\mathbb{Q}_p (where k is a corresponding unramified analytic field which is discretely-valued and the corresponding residue field is finite). Then we have the following:

²This is the corresponding derived version of pseudocoherence from [Lu1], [Lu2].

Proposition 2.1.4. Consider the corresponding projective map $g : X_{\text{pro-étale}} \rightarrow X_{\text{ét}}$ and the projective map $f : X_{\text{pro-étale}} \rightarrow X$. Then we have the following two strictly exact long exact sequences ³:

$$0 \longrightarrow \text{Kan}_{\text{Left}} \text{deRham}_{k[\mathcal{O}]}^{\text{degreenumber}} / k, \text{topo}, Z \xrightarrow{\partial} \text{Kan}_{\text{Left}} \text{deRham}_{X[\mathcal{O}]}^{\text{degreenumber}} / X, \text{topo}, Z \xrightarrow{\partial} \text{Kan}_{\text{Left}} \text{deRham}_{X[\mathcal{O}]}^{\text{degreenumber}} \otimes f^{-1} \text{deRham}_{X, \text{topo}}^1 \longrightarrow \dots$$

and

$$0 \longrightarrow \text{Kan}_{\text{Left}} \text{deRham}_{k[\mathcal{O}]}^{\text{degreenumber}} / k, \text{topo}, Z \xrightarrow{\partial} \text{Kan}_{\text{Left}} \text{deRham}_{X[\mathcal{O}]}^{\text{degreenumber}} / X_{\text{ét}}, \text{topo}, Z \xrightarrow{\partial} \text{Kan}_{\text{Left}} \text{deRham}_{X[\mathcal{O}]}^{\text{degreenumber}} \otimes f^{-1} \text{deRham}_{X_{\text{ét}}, \text{topo}}^1 \longrightarrow \dots$$

Proof. This is actually a direct consequence [GL, Theorem 1.2]. □

Proposition 2.1.5. Consider the corresponding projective map $g : X_{\text{pro-étale}} \rightarrow X_{\text{ét}}$ and the projective map $f : X_{\text{pro-étale}} \rightarrow X$. Let M be a corresponding Z -projective differential crystal spectrum. Then we have the following two strictly exact long exact sequences ⁴:

$$M^{\otimes \mathbb{L}}(0) \longrightarrow \text{deRham}_{k[\mathcal{O}]}^{\text{degreenumber}} / k, \text{topo}, Z \xrightarrow{\partial} \text{deRham}_{X[\mathcal{O}]}^{\text{degreenumber}} / X, \text{topo}, Z \xrightarrow{\partial} \text{deRham}_{X[\mathcal{O}]}^{\text{degreenumber}} \otimes f^{-1} \text{deRham}_{X, \text{topo}}^1 \longrightarrow \dots,$$

and

$$M^{\otimes \mathbb{L}}(0) \longrightarrow \text{deRham}_{k[\mathcal{O}]}^{\text{degreenumber}} / k, \text{topo}, Z \xrightarrow{\partial} \text{deRham}_{X[\mathcal{O}]}^{\text{degreenumber}} / X_{\text{ét}}, \text{topo}, Z \xrightarrow{\partial} \text{deRham}_{X[\mathcal{O}]}^{\text{degreenumber}} \otimes f^{-1} \text{deRham}_{X_{\text{ét}}, \text{topo}}^1 \longrightarrow \dots.$$

Proof. This is actually a direct consequence of the previous proposition due to the corresponding flatness. □

³We should have the corresponding naturality taking into the following form:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Kan}_{\text{Left}} \text{deRham}_{k[\mathcal{O}]}^{\text{degreenumber}} / k, \text{topo}, Z & \xrightarrow{\partial} & \text{Kan}_{\text{Left}} \text{deRham}_{X[\mathcal{O}]}^{\text{degreenumber}} / X, \text{topo}, Z & \xrightarrow{\partial} & \text{Kan}_{\text{Left}} \text{deRham}_{X[\mathcal{O}]}^{\text{degreenumber}} \otimes f^{-1} \text{deRham}_{X, \text{topo}}^1 \longrightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Kan}_{\text{Left}} \text{deRham}_{k[\mathcal{O}]}^{\text{degreenumber}} / k, \text{topo}, Z & \xrightarrow{\partial} & \text{Kan}_{\text{Left}} \text{deRham}_{X[\mathcal{O}]}^{\text{degreenumber}} / X_{\text{ét}}, \text{topo}, Z & \xrightarrow{\partial} & \text{Kan}_{\text{Left}} \text{deRham}_{X[\mathcal{O}]}^{\text{degreenumber}} \otimes f^{-1} \text{deRham}_{X_{\text{ét}}, \text{topo}}^1 \longrightarrow \dots \end{array}$$

⁴Here we drop the notation Kan_{Left} .

2.1.2 Analytic André-Quillen Homology and Analytic Derived de Rham complex of Pseudorigid Spaces

We now consider the analytic André-Quillen Homology and analytic Derived de Rham complex of pseudorigid space, which are very crucial in some development in [Bel1] in the arithmetic family. Therefore we just investigate the corresponding picture in the corresponding geometric family.

Setting 2.1.6. We consider now a corresponding morphism taking the corresponding form of $A \rightarrow B$ where A is going to be a pseudorigid affinoid algebra over \mathbb{Z}_p , and B is going to be a perfectoid chart of A in the corresponding pro-étale site of the pseudorigid affinoid space attached to A . As in [Bel1, Definition 3.1, and below Definition 3.1] in our situation A is of topologically finite type over $\mathcal{O}_K[[t]] \langle \pi^a / t^b \rangle [1/t]$, where K is a discrete valued field containing \mathbb{Q}_p and $(a, b) = 1$ ⁵.

From [Bel1, Definition 3.1, and below Definition 3.1] we have the following:

Lemma 2.1.7. *We have the following statements:*

- A. *A as above is Tate, complete over \mathcal{O}_K ;*
- B. *The ring A has a ring of definition A_0 which is of \mathcal{O}_K -formally finite type;*
- C. *The ring A_0 is of topologically finite type over $\mathcal{O}_K[[t]] \langle \pi^a / t^b \rangle$.*

Proof. It is very easy to see that the corresponding results hold in our current situation over \mathcal{O}_K . □

As in the corresponding construction in the rigid situation in the previous section following [An1, Chapitre 3], [An2], [B1, Chapter 2, Chapter 8], [Bei, Chapter 1], [G1, Chapter 5], [GL, Chapter 3, Chapter 4], [Ill1, Chapitre II, Chapitre III], [Ill2, Chapitre VIII], [Qui, Section 4] we give the following definitions. First we consider the following new setting:

Setting 2.1.8. Now as in the above notion we will consider a general map of rings $A \rightarrow B$ over $\mathcal{O}_K[[t]] \langle \pi^a / t^b \rangle [1/t]$ where we have the corresponding map of the associated ring of definitions $A_0 \rightarrow B_0$ over $\mathcal{O}_K[[t]] \langle \pi^a / t^b \rangle$ such that A_0, B_0 are basically I -adic (where $\mathcal{O}_K[[t]] \langle \pi^a / t^b \rangle$ is I -adic).

Definition 2.1.9. We start from the corresponding construction of the algebraic p -adic derived de Rham complex for a map $A \rightarrow B$. Fix a pair of ring of definitions A_0, B_0 in A, B respectively. Then this is the corresponding derived differential complex attached to the polynomial resolution of B_0 :

$$A_0[A_0[B_0]] \dots, \tag{2.1.22}$$

which is now denoted by $\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0, \text{alg}}^{\text{degreenumber}}$ after taking the corresponding left Kan extension. The corresponding cotangent complex associated is defined to be just:

$$\mathbb{L}_{B_0/A_0, \text{alg}} := \text{deRham}_{A_0[B_0]^{\text{degreenumber}}/A_0, \text{alg}}^1 \otimes_{A_0[B_0]^{\text{degreenumber}}} B_0. \tag{2.1.23}$$

The corresponding algebraic André-Quillen homologies are defined to be:

$$H_{\text{degreenumber}, \text{AQ,alg}} := \pi_{\text{degreenumber}}(\mathbb{L}_{B_0/A_0, \text{alg}}). \tag{2.1.24}$$

⁵Certainly one can also consider the characteristic p situation.

The corresponding topological André-Quillen complex is actually the complete version of the corresponding algebraic ones above by considering the corresponding derived I -completion over the simplicial module structure:

$$\mathbb{L}_{B_0/A_0, \text{topo}} := R \varprojlim_I \text{Kos}_I \left((\text{deRham}_{A_0[B_0]^{\text{degreenumber}}/A_0, \text{alg}}^1 \otimes_{A_0[B_0]^{\text{degreenumber}}} B_0) \right). \quad (2.1.25)$$

Then we consider the corresponding derived algebraic de Rham complex which is just defined to be:

$$\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0, \text{alg}}^{\text{degreenumber}}, \text{Kan}_{\text{Left}} \text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0, \text{alg}}^1}^*. \quad (2.1.26)$$

We then take the corresponding derived I -completion and we denote that by:

$$\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0, \text{topo}}^{\text{degreenumber}} := R \varprojlim_I \text{Kos}_I \left(\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0, \text{alg}}^{\text{degreenumber}}, \text{Kan}_{\text{Left}} \text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0, \text{alg}}^{\text{degreenumber}}}^* \right), \quad (2.1.27)$$

$$\text{Fil}_{\text{deRham}_{B_0/A_0, \text{topo}}^{\text{degreenumber}}}^* := R \varprojlim_I \text{Kos}_I \left(\text{Kan}_{\text{Left}} \text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0, \text{alg}}^{\text{degreenumber}}}^* \right). \quad (2.1.28)$$

Then we consider the following construction for the map $A \rightarrow B$ by putting:

$$\mathbb{L}_{B_0/A_0, \text{topo}} := \text{Colim}_{A_0 \rightarrow B_0} \mathbb{L}_{B_0/A_0, \text{topo}}[1/t], \quad (2.1.29)$$

$$H_{\text{degreenumber}, \text{AQ}, \text{topo}} := \pi_{\text{degreenumber}}(\mathbb{L}_{B/A, \text{topo}}), \quad (2.1.30)$$

$$\text{Kan}_{\text{Left}} \text{deRham}_{B/A, \text{topo}}^{\text{degreenumber}} := \text{Colim}_{A_0 \rightarrow B_0} \text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0, \text{topo}}^{\text{degreenumber}}[1/t], \quad (2.1.31)$$

$$\text{Kan}_{\text{Left}} \text{Fil}_{\text{deRham}_{B/A, \text{topo}}^{\text{degreenumber}}}^* := \text{Colim}_{A_0 \rightarrow B_0} \text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0, \text{topo}}^{\text{degreenumber}}}^*[1/t]. \quad (2.1.32)$$

Then we need to take the corresponding Hodge-Filtered completion by using the corresponding filtration associated as above to achieve the corresponding Hodge-complete objects in the corresponding filtered ∞ -categories:

$$\text{Kan}_{\text{Left}} \text{deRham}_{B/A, \text{topo}, \text{Hodge}}^{\text{degreenumber}}, \text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{B/A, \text{topo}, \text{Hodge}}^{\text{degreenumber}}}^*. \quad (2.1.33)$$

Definition 2.1.10. We define the corresponding finite projective filtered crystals to be the corresponding finite projective module spectra over the topological filtered E_∞ -ring $\text{Kan}_{\text{Left}} \text{deRham}_{B/A, \text{topo}}^{\text{degreenumber}}$ with the corresponding induced filtrations.

Definition 2.1.11. We define the corresponding almost perfect⁶ filtered crystals to be the corresponding almost perfect module spectra over the topological filtered E_∞ -ring $\text{Kan}_{\text{Left}} \text{deRham}_{B/A, \text{topo}}^{\text{degreenumber}}$ with the corresponding induced filtrations.

We now consider the corresponding large coefficient local systems over pseudorigid spaces where Z now is a topological algebra over \mathbb{Z}_p ⁷.

⁶This is the corresponding derived version of pseudocoherence from [Lu1], [Lu2].

⁷It is better to assume that it is basically completely flat.

Definition 2.1.12. We define the following Z deformed version of the corresponding complete version of the corresponding André-Quillen homology and the corresponding complete version of the corresponding derived de Rham complex. We start from the corresponding construction of the algebraic p -adic derived de Rham complex for a map $A \rightarrow B$. Fix a pair of ring of definitions A_0, B_0 in A, B respectively. Then this is the corresponding derived differential complex attached to the polynomial resolution of B_0 :

$$A_0[A_0[B_0]]\dots, \quad (2.1.34)$$

which is now denoted by $\text{deRham}_{A_0[B_0]^{\text{degreenumber}}/A_0, \text{alg}}^{\text{degreenumber}}$. The corresponding cotangent complex associated is defined to be just:

$$\mathbb{L}_{B_0/A_0, \text{alg}} := \text{deRham}_{A_0[B_0]^{\text{degreenumber}}/A_0, \text{alg}}^1 \otimes_{A_0[B_0]^{\text{degreenumber}}} B_0. \quad (2.1.35)$$

The corresponding algebraic André-Quillen homologies are defined to be:

$$H_{\text{degreenumber}, \text{AQ}, \text{alg}} := \pi_{\text{degreenumber}}(\mathbb{L}_{B_0/A_0, \text{alg}}). \quad (2.1.36)$$

The corresponding topological André-Quillen complex is actually the complete version of the corresponding algebraic ones above by considering the corresponding derived I -completion over the simplicial module structure:

$$\mathbb{L}_{B_0/A_0, \text{topo}} := R \varprojlim_k \text{Kos}_I \left((\text{deRham}_{A_0[B_0]^{\text{degreenumber}}/A_0, \text{alg}}^1 \otimes_{A_0[B_0]^{\text{degreenumber}}} B_0) \right). \quad (2.1.37)$$

Taking the product with Z we have the corresponding integral version of the topological André-Quillen complex⁸:

$$\mathbb{L}_{B_0/A_0, \text{topo}, Z} := \mathbb{L}_{B_0/A_0, \text{topo}} \otimes_{\mathbb{Z}_p} Z \quad (2.1.38)$$

Then we consider the corresponding derived algebraic de Rham complex which is just defined to be:

$$\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0, \text{alg}}^{\text{degreenumber}}, \text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0, \text{alg}}^{\text{degreenumber}}}^* \quad (2.1.39)$$

We then take the corresponding derived I -completion and we denote that by:

$$\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0, \text{topo}}^{\text{degreenumber}} := R \varprojlim_k \text{Kos}_I \left(\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0, \text{alg}}^{\text{degreenumber}}, \text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0, \text{alg}}^{\text{degreenumber}}}^* \right), \quad (2.1.40)$$

$$\text{Kan}_{\text{Left}} \text{Fil}_{\text{deRham}_{B_0/A_0, \text{topo}}^{\text{degreenumber}}}^* := R \varprojlim_k \text{Kos}_I \left(\text{Kan}_{\text{Left}} \text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0, \text{alg}}^{\text{degreenumber}}}^* \right). \quad (2.1.41)$$

Before considering the corresponding integral version we just consider the corresponding product of these \mathbb{E}_∞ -rings with Z to get⁹:

$$\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0, \text{topo}, Z}^{\text{degreenumber}} := \text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0, \text{topo}}^{\text{degreenumber}} \otimes_{\mathbb{Z}_p} Z, \quad (2.1.42)$$

$$\text{Kan}_{\text{Left}} \text{Fil}_{\text{deRham}_{B_0/A_0, \text{topo}, Z}^{\text{degreenumber}}}^* := \text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0, \text{topo}}^{\text{degreenumber}}}^* \otimes_{\mathbb{Z}_p} Z. \quad (2.1.43)$$

⁸Here we did not take the corresponding derived completion, but in some situation this is achievable.

⁹Here again we did not take the corresponding derived completion, but in some situation this is achievable, for instance when the ring Z is endowed with derived J -complete topology.

Then we consider the following construction for the map $A \rightarrow B$ by putting:

$$\mathbb{L}_{B_0/A_0, \text{topo}, Z} := \text{Colim}_{A_0 \rightarrow B_0} \mathbb{L}_{B_0/A_0, \text{topo}, Z}[1/t], \quad (2.1.44)$$

$$H_{\text{degreenumber}, \text{AQ}, \text{topo}, Z} := \pi_{\text{degreenumber}}(\mathbb{L}_{B/A, \text{topo}, Z}), \quad (2.1.45)$$

$$\text{Kan}_{\text{Left}} \text{deRham}_{B/A, \text{topo}, Z}^{\text{degreenumber}} := \text{Colim}_{A_0 \rightarrow B_0} \text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0, \text{topo}, Z}^{\text{degreenumber}}[1/t], \quad (2.1.46)$$

$$\text{Kan}_{\text{Left}} \text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{B/A, \text{topo}, Z}^{\text{degreenumber}}}^* := \text{Colim}_{A_0 \rightarrow B_0} \text{Kan}_{\text{Left}} \text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0, \text{topo}, Z}^1}^*[1/t]. \quad (2.1.47)$$

Then we need to take the corresponding Hodge-Filtered completion by using the corresponding filtration associated as above to achieve the corresponding Hodge-complete objects in the corresponding filtered ∞ -categories:

$$\text{Kan}_{\text{Left}} \text{deRham}_{B/A, \text{topo}, \text{Hodge}, Z}^{\text{degreenumber}}, \text{Kan}_{\text{Left}} \text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{B/A, \text{topo}, \text{Hodge}, Z}^1}^*. \quad (2.1.48)$$

Example 2.1.13. Now we construct the corresponding pseudorigid analog of the corresponding example in [GL, Example 4.7]. Now we consider the corresponding the map:

$$\mathbb{Z}_p[[u]] \left\langle \frac{p^a}{u^b} \right\rangle [1/u] \langle X_1^\pm, X_2^\pm, \dots, X_d^\pm \rangle \longrightarrow \mathbb{Z}_p[[u]] \left\langle \frac{p^a}{u^b} \right\rangle [1/u] \langle X_1^{\pm/p^\infty}, X_2^{\pm/p^\infty}, \dots, X_d^{\pm/p^\infty} \rangle \quad (2.1.49)$$

which could be written as:

$$\mathbb{Z}_p[[u]] \left\langle \frac{p^a}{u^b} \right\rangle [1/u] \langle X_1^\pm, X_2^\pm, \dots, X_d^\pm \rangle \longrightarrow \quad (2.1.50)$$

$$\mathbb{Z}_p[[u]] \left\langle \frac{p^a}{u^b} \right\rangle [1/u] \langle X_1^\pm, X_2^\pm, \dots, X_d^\pm \rangle \langle Y_1^{\pm/p^\infty}, Y_2^{\pm/p^\infty}, \dots, Y_d^{\pm/p^\infty} \rangle / (X_i - Y_i, i = 1, \dots, d). \quad (2.1.51)$$

So in our situation the corresponding Hodge complete topological derived de Rham complex will be just:

$$\mathbb{Z}_p[[u]] \left\langle \frac{p^a}{u^b} \right\rangle [1/u] \langle Y_1^{\pm/p^\infty}, Y_2^{\pm/p^\infty}, \dots, Y_d^{\pm/p^\infty} \rangle [[Z_1, \dots, Z_d, Z_i = X_i - Y_i, i = 1, \dots, d]]. \quad (2.1.52)$$

We now follow the corresponding previous works [B1, Chapter 2, Chapter 8], [Bei, Chapter 1], [G1, Chapter 5], [GL, Chapter 3, Chapter 4], [Ill1], [Ill2, Chapitre VIII] to study the corresponding functoriality for some triple $A \rightarrow B \rightarrow C$ where we have first in the algebraic setting the following result. Here we first consider more general setting where we will consider those rings over $\mathcal{O}_K[[t]] \left\langle \frac{p^a}{t^b} \right\rangle$ which is t -adically complete.

Proposition 2.1.14. *The corresponding functoriality for the corresponding algebraic derived de Rham complex holds in our situation for the corresponding integral adic rings A, B, C in the context of this section, namely we have the following corresponding commutative diagram:*

$$\begin{array}{ccc} \text{Kan}_{\text{Left}} \text{deRham}_{B/A, \text{alg}}^{\text{degreenumber}} & \longrightarrow & \text{Kan}_{\text{Left}} \text{deRham}_{B/A, \text{alg}}^{\text{degreenumber}} \\ \downarrow & & \downarrow \\ B & \longrightarrow & \text{Kan}_{\text{Left}} \text{deRham}_{B/A, \text{alg}}^{\text{degreenumber}} \end{array}$$

Proof. See [GL, Lemma 3.3]. □

Proposition 2.1.15. *The corresponding functoriality for the corresponding algebraic derived de Rham complex holds in our situation for the corresponding integral adic rings A, B, C in the context of this section, namely we have the following corresponding commutative diagram:*

$$\begin{array}{ccc}
 \text{Kan}_{\text{Left}}\text{deRham}_{B/A, \text{topo}, \text{Hodge}}^{\text{degreenumber}} & \longrightarrow & \text{Kan}_{\text{Left}}\text{deRham}_{B/A, \text{topo}, \text{Hodge}}^{\text{degreenumber}} \\
 \downarrow & & \downarrow \\
 B & \longrightarrow & \text{Kan}_{\text{Left}}\text{deRham}_{B/A, \text{topo}, \text{Hodge}}^{\text{degreenumber}}
 \end{array}$$

Proof. Just take the corresponding derived I -completion of the diagram in the previous proposition. □

The following conjectures are literally inspired by the rigid analytic situation in [GL, Theorem 1.2]. We use the notation $*$ to denote the ring $\mathcal{O}_K[[t]] \langle \pi^a / t^b \rangle [1/t]$. We assume the spaces in the following two conjectures are smooth pseudo-rigid analytic spaces over $\mathcal{O}_K[[t]] \langle \pi^a / t^b \rangle [1/t]$.

Conjecture 2.1.16. *The construction in this section could be carried over the pro-étale sites ¹⁰. Consider the corresponding projective map $g : X_{\text{pro-étale}} \rightarrow X_{\text{ét}}$ and the projective map $f : X_{\text{pro-étale}} \rightarrow X$. Then we have the following two strictly exact long exact sequences ¹¹:*

$$0 \longrightarrow \text{deRham}_{*[\mathbb{C}]}^{\text{degree number}} / \mathbb{A}, \text{topo}, \mathbb{Z} \xrightarrow{\partial} \text{deRham}_{X(\mathbb{C})}^{\text{degree number}} / X, \text{topo}, \mathbb{Z} \xrightarrow{\partial} \text{deRham}_{X(\mathbb{C})}^{\text{degree number}} / X, \text{topo}, \mathbb{Z} \otimes f^{-1} \text{deRham}_{X_{\text{ét}}, \text{topo}}^{\partial} \longrightarrow \dots,$$

and

$$0 \longrightarrow \text{deRham}_{*[\mathbb{C}]}^{\text{degree number}} / \mathbb{A}, \text{topo}, \mathbb{Z} \xrightarrow{\partial} \text{deRham}_{X(\mathbb{C})}^{\text{degree number}} / X_{\text{ét}}, \text{topo}, \mathbb{Z} \xrightarrow{\partial} \text{deRham}_{X(\mathbb{C})}^{\text{degree number}} / X, \text{topo}, \mathbb{Z} \otimes f^{-1} \text{deRham}_{X_{\text{ét}}, \text{topo}}^{\partial} \longrightarrow \dots$$

Conjecture 2.1.17. *Consider the corresponding projective map $g : X_{\text{pro-étale}} \rightarrow X_{\text{ét}}$ and the projective map $f : X_{\text{pro-étale}} \rightarrow X$. Let M be a corresponding \mathbb{Z} -projective differential crystal spectrum. Then we have the following two strictly exact long exact sequences:*

$$M \otimes^{\mathbb{L}} 0 \longrightarrow \text{deRham}_{\mathcal{O}_k[\mathbb{C}]}^{\text{degree number}} / \mathcal{O}_k, \text{topo}, \mathbb{Z} \xrightarrow{\partial} \text{deRham}_{X(\mathbb{C})}^{\text{degree number}} / X, \text{topo}, \mathbb{Z} \xrightarrow{\partial} \text{deRham}_{X(\mathbb{C})}^{\text{degree number}} / X, \text{topo}, \mathbb{Z} \otimes f^{-1} \text{deRham}_{X_{\text{ét}}, \text{topo}}^{\partial} \longrightarrow \dots,$$

and

$$M \otimes^{\mathbb{L}} 0 \longrightarrow \text{deRham}_{\mathcal{O}_k[\mathbb{C}]}^{\text{degree number}} / \mathcal{O}_k, \text{topo}, \mathbb{Z} \xrightarrow{\partial} \text{deRham}_{X(\mathbb{C})}^{\text{degree number}} / X_{\text{ét}}, \text{topo}, \mathbb{Z} \xrightarrow{\partial} \text{deRham}_{X(\mathbb{C})}^{\text{degree number}} / X, \text{topo}, \mathbb{Z} \otimes f^{-1} \text{deRham}_{X_{\text{ét}}, \text{topo}}^{\partial} \longrightarrow \dots.$$

¹⁰For more on the foundations here for the pro-étale sites see [Ked1, Theorem 2.9.9, Remark 2.9.10]

¹¹We should have the corresponding naturality taking into the following form:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{deRham}_{*[\mathbb{C}]}^{\text{degree number}} / \mathbb{A}, \text{topo}, \mathbb{Z} & \xrightarrow{\partial} & \text{deRham}_{X(\mathbb{C})}^{\text{degree number}} / X, \text{topo}, \mathbb{Z} & \xrightarrow{\partial} & \text{deRham}_{X(\mathbb{C})}^{\text{degree number}} / X, \text{topo}, \mathbb{Z} \otimes f^{-1} \text{deRham}_{X_{\text{ét}}, \text{topo}}^{\partial} \longrightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{deRham}_{*[\mathbb{C}]}^{\text{degree number}} / \mathbb{A}, \text{topo}, \mathbb{Z} & \xrightarrow{\partial} & \text{deRham}_{X(\mathbb{C})}^{\text{degree number}} / X_{\text{ét}}, \text{topo}, \mathbb{Z} & \xrightarrow{\partial} & \text{deRham}_{X(\mathbb{C})}^{\text{degree number}} / X, \text{topo}, \mathbb{Z} \otimes f^{-1} \text{deRham}_{X_{\text{ét}}, \text{topo}}^{\partial} \longrightarrow \dots \end{array}$$

2.1.3 Derived (p, I) -Complete Derived de Rham complex of Derived Adic Rings

The corresponding construction of the derived de Rham complex could be defined for general derived spaces. Along our discussion in the situations of rigid analytic spaces and the corresponding pseudorigid spaces we now focus on the corresponding derived adic rings.

Setting 2.1.18. We now fix a bounded morphism of simplicial adic rings $A \rightarrow B$ over A^* where A^* contains a corresponding ring of definition A_0^* which is derived complete with respect to the (p, I) -topology and we assume that A is adic and we assume that (A_0^*, I) is a prism in [BS1] namely we at least require that the corresponding δ -structure on the corresponding ring will induce the map $\varphi(\cdot) := \cdot^p + p\delta(\cdot)$ such that we have the situation where $p \in (I, \varphi(I))$. For A or B respectively we assume this contains a subring A_0 or B_0 (over A_0^*) respectively such that we have A_0 or B_0 respectively is derived complete with respect to the corresponding derived (p, I) -topology and we assume that $B = B_0[1/f, f \in I]$ (same for A). All the adic rings are assumed to be open mapping. We use the notation d to denote a corresponding primitive element as in [BS1, Section 2.3] for A^* . We are going to assume that p is a topologically nilpotent element.

As in the corresponding construction in the rigid and pseudocoherent situation in the previous section following [An1, Chapitre 3], [An2], [B1, Chapter 2, Chapter 8], [Bei, Chapter 1], [G1, Chapter 5], [GL, Chapter 3, Chapter 4], [Ill1, Chapitre II, Chapitre III], [Ill2, Chapitre VIII], [Qui, Section 4], we have the following:

Definition 2.1.19. We start from the corresponding construction of the algebraic p -adic derived de Rham complex for a map $A \rightarrow B$. Fix a pair of ring of definitions A_0, B_0 in A, B respectively. Then this is the corresponding left Kan extended derived differential complex attached to B_0/A_0 which is now denoted by $\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0, \text{alg}}^{\text{degreenumber}12}$ from the corresponding (p, I) -complete commutative rings. The corresponding cotangent complex associated is defined to be just:

$$\mathbb{L}_{B_0/A_0, \text{alg}} := \text{deRham}_{P_{B_0}^{\text{degreenumber}}/A_0, \text{alg}}^1 \otimes_{P_{B_0}^{\text{degreenumber}}} B_0. \quad (2.1.53)$$

The corresponding algebraic André-Quillen homologies are defined to be:

$$H_{\text{degreenumber}, \text{AQ}, \text{alg}} := \pi_{\text{degreenumber}}(\mathbb{L}_{B_0/A_0, \text{alg}}). \quad (2.1.54)$$

The corresponding topological André-Quillen complex is actually the complete version of the corresponding algebraic ones above by considering the corresponding derived (p, I) -completion over the simplicial module structure:

$$\mathbb{L}_{B_0/A_0, \text{topo}} := R \varprojlim_{(p, I)} \text{Kos}_{(p, I)} \left(\left(\text{deRham}_{P_{B_0}^{\text{degreenumber}}/A_0, \text{alg}}^1 \otimes_{P_{B_0}^{\text{degreenumber}}} B_0 \right) \right). \quad (2.1.55)$$

Then we consider the corresponding derived algebraic de Rham complex which is just defined to be:

$$\text{Kan}_{\text{Left}} \text{deRham}_{-/A_0, \text{alg}}^{\text{degreenumber}}(B_0), \text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0, \text{alg}}^{\text{degreenumber}}}^* \quad (2.1.56)$$

¹²Namely over A_0 we take the suitable corresponding left Kan extension, then apply to B_0 .

We then take the corresponding derived (p, I) -completion and we denote that by:

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{B_0/A_0,\mathrm{topo}}^{\mathrm{degree\,number}} := R\varprojlim_{(p,I)} \mathrm{Kos}_{(p,I)} \left(\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{B_0/A_0,\mathrm{alg}}^{\mathrm{degree\,number}}, \mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{B_0/A_0,\mathrm{alg}}}^* \right), \quad (2.1.57)$$

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{B_0/A_0,\mathrm{topo}}^{\mathrm{degree\,number}}}^* := R\varprojlim_{(p,I)} \mathrm{Kos}_{(p,I)} \left(\mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{B_0/A_0,\mathrm{alg}}}^* \right). \quad (2.1.58)$$

Then in the situation that all the rings are classical adic rings we consider the following construction for the map $A \rightarrow B$ by putting:

$$\mathbb{L}_{B/A,\mathrm{topo}} := \mathrm{Colim}_{A_0 \rightarrow B_0} \mathbb{L}_{B_0/A_0,\mathrm{topo}}[1/(d)], \quad (2.1.59)$$

$$H^{\mathrm{degree\,number},\mathrm{AQ},\mathrm{topo}} := \pi^{\mathrm{degree\,number}}(\mathbb{L}_{B/A,\mathrm{topo}}), \quad (2.1.60)$$

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{B/A,\mathrm{topo}}^{\mathrm{degree\,number}} := \mathrm{Colim}_{A_0 \rightarrow B_0} \mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{B_0/A_0,\mathrm{topo}}^{\mathrm{degree\,number}}[1/(d)], \quad (2.1.61)$$

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{B/A,\mathrm{topo}}^{\mathrm{degree\,number}}}^* := \mathrm{Colim}_{A_0 \rightarrow B_0} \mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{B_0/A_0,\mathrm{topo}}^{\mathrm{degree\,number}}}^*[1/(d)]. \quad (2.1.62)$$

Then we need to take the corresponding Hodge-Filtered completion by using the corresponding filtration associated as above to achieve the corresponding Hodge-complete objects in the corresponding filtered ∞ -categories:

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{B/A,\mathrm{topo},\mathrm{Hodge}}^{\mathrm{degree\,number}}, \mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{B/A,\mathrm{topo},\mathrm{Hodge}}^{\mathrm{degree\,number}}}^* \quad (2.1.63)$$

Definition 2.1.20. We define the corresponding finite projective filtered crystals to be the corresponding finite projective module spectra over the topological filtered E_∞ -ring $\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{B/A,\mathrm{topo}}^{\mathrm{degree\,number}}$ with the corresponding induced filtrations.

Definition 2.1.21. We define the corresponding almost perfect¹³ filtered crystals to be the corresponding almost perfect module spectra over the topological filtered E_∞ -ring $\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{B/A,\mathrm{topo}}^{\mathrm{degree\,number}}$ with the corresponding induced filtrations.

We now consider the corresponding large coefficient local systems over pseudorigid spaces where Z now is a simplicial topological algebra over \mathbb{Z}_p ¹⁴.

Definition 2.1.22. We define the following Z deformed version of the corresponding complete version of the corresponding André-Quillen homology and the corresponding complete version of the corresponding derived de Rham complex. We start from the corresponding construction of the algebraic p -adic derived de Rham complex for a map $A \rightarrow B$. Fix a pair of ring of definitions A_0, B_0 in A, B respectively. Then this is the corresponding left Kan extended derived differential complex attached to B_0 from the corresponding derived (p, I) -complete commutative rings which is now denoted by $\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{B_0/A_0,\mathrm{alg}}^{\mathrm{degree\,number}}$. The corresponding cotangent complex associated is defined to be just:

$$\mathbb{L}_{B_0/A_0,\mathrm{alg}} := \mathrm{deRham}_{P_{B_0}^{\mathrm{degree\,number}}/A_0,\mathrm{alg}}^1 \otimes_{P_{B_0}^{\mathrm{degree\,number}}} B_0. \quad (2.1.64)$$

¹³This is the corresponding derived version of pseudocoherence from [Lu1], [Lu2].

¹⁴It is better to assume that it is basically derived completely flat.

The corresponding algebraic André-Quillen homologies are defined to be:

$$H_{\text{degreenumber,AQ,alg}} := \pi_{\text{degreenumber}}(\mathbb{L}_{B_0/A_0,\text{alg}}). \quad (2.1.65)$$

The corresponding topological André-Quillen complex is actually the complete version of the corresponding algebraic ones above by considering the corresponding derived (p, I) -completion over the simplicial module structure:

$$\mathbb{L}_{B_0/A_0,\text{topo}} := R\varprojlim_{\leftarrow k} \text{Kos}_{(p,I)} \left((\text{deRham}_{P_{B_0}}^{\text{degreenumber}/A_0,\text{alg}} \otimes_{P_{B_0}^{\text{degreenumber}}} B_0) \right). \quad (2.1.66)$$

Taking the product with Z we have the corresponding integral version of the topological André-Quillen complex:

$$\mathbb{L}_{B_0/A_0,\text{topo},Z} := \mathbb{L}_{B_0/A_0,\text{topo}} \otimes_{\mathbb{Z}_p} Z \quad (2.1.67)$$

Then we consider the corresponding derived algebraic de Rham complex which is just defined to be:

$$\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0,\text{alg}}^{\text{degreenumber}}, \text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0,\text{alg}}^{\text{degreenumber}}}^* \quad (2.1.68)$$

We then take the corresponding derived (p, I) -completion and we denote that by:

$$\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0,\text{topo}}^{\text{degreenumber}} := R\varprojlim_{\leftarrow k} \text{Kos}_{(p,I)} \left(\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0,\text{alg}}^{\text{degreenumber}}, \text{Fil}_{\text{deRham}_{B_0/A_0,\text{alg}}^{\text{degreenumber}}}^* \right), \quad (2.1.69)$$

$$\text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0,\text{topo}}^{\text{degreenumber}}}^* := R\varprojlim_{\leftarrow k} \left(\text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0,\text{alg}}^{\text{degreenumber}}}^* \right). \quad (2.1.70)$$

Before considering the corresponding integral version we just consider the corresponding product of these \mathbb{E}_∞ -rings with Z to get:

$$\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0,\text{topo},Z}^{\text{degreenumber}} := \text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0,\text{topo}}^{\text{degreenumber}} \otimes_{\mathbb{Z}_p} Z, \quad (2.1.71)$$

$$\text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0,\text{topo},Z}^{\text{degreenumber}}}^* := \text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0,\text{topo}}^{\text{degreenumber}}}^* \otimes_{\mathbb{Z}_p} Z. \quad (2.1.72)$$

When we have that the corresponding ring Z is also derived (p, I) -topologized and commutative, then we can further take the corresponding derived (p, I) -completion to achieve the corresponding derived completed version:

$$\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0,\text{topo},Z}^{\text{degreenumber}} := \text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0,\text{topo}}^{\text{degreenumber}} \widehat{\otimes}_{\mathbb{Z}_p} Z, \quad (2.1.73)$$

$$\text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0,\text{topo},Z}^{\text{degreenumber}}}^* := \text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0,\text{topo}}^{\text{degreenumber}}}^* \widehat{\otimes}_{\mathbb{Z}_p} Z. \quad (2.1.74)$$

Then in the situation that all the rings are classical adic rings we consider the following construction for the map $A \rightarrow B$ by putting:

$$\mathbb{L}_{B/A,\text{topo},Z} := \text{Colim}_{A_0 \rightarrow B_0} \mathbb{L}_{B_0/A_0,\text{topo},Z} [1/(d)], \quad (2.1.75)$$

$$H_{\text{degreenumber,AQ,topo},Z} := \pi_{\text{degreenumber}}(\mathbb{L}_{B/A,\text{topo},Z}), \quad (2.1.76)$$

$$\text{Kan}_{\text{Left}} \text{deRham}_{B/A,\text{topo},Z}^{\text{degreenumber}} := \text{Colim}_{A_0 \rightarrow B_0} \text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0,\text{topo},Z}^{\text{degreenumber}} [1/(d)], \quad (2.1.77)$$

$$\text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{B/A,\text{topo},Z}^{\text{degreenumber}}}^* := \text{Colim}_{A_0 \rightarrow B_0} \text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{B_0/A_0,\text{topo},Z}^{\text{degreenumber}}}^* [1/(d)]. \quad (2.1.78)$$

Then we need to take the corresponding Hodge-Filtered completion by using the corresponding filtration associated as above to achieve the corresponding Hodge-complete objects in the corresponding filtered ∞ -categories:

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{B/A,\mathrm{topo},\mathrm{Hodge},Z}^{\mathrm{degreenumber}} \mathrm{Fil}^*_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{B/A,\mathrm{topo},\mathrm{Hodge},Z}^{\mathrm{degreenumber}}} \quad (2.1.79)$$

2.2 Topological Logarithmic Derived De Rham Complexes

2.2.1 Logarithmic Setting for Rigid Analytic Spaces

In this section we are now going to follow Gabber's construction as in [O, Chapter 8] and the extension by [B1, Chapter 5, Chapter 6, Chapter 7] to consider the corresponding construction of the topological complete logarithmic cotangent complexes and the corresponding topological logarithmic derived de Rham complex following the corresponding construction for rigid spaces. Certainly the corresponding construction will also follow closely the corresponding [An1, Chapitre 3], [An2], [B1, Chapter 2, Chapter 8], [Bei, Chapter 1], [G1, Chapter 5], [GL, Chapter 3, Chapter 4], [Ill1, Chapitre II, Chapitre III], [Ill2, Chapitre VIII], [Qui, Section 4] as if we do not have the corresponding log structures, in particular the geometry context underlying is just as in [GL, Chapter 3, Chapter 4]. For the geometric foundation of log adic spaces, see [DLLZ1]. We start from the corresponding construction of the algebraic p -adic logarithmic derived de Rham complex for a map $(A, M) \rightarrow (B, N)$ of p -complete rings carrying the corresponding log structures, here we use the corresponding notation $(*, ?)$, $*$ = A, B to denote the corresponding admissible log rings where $?$ represents the corresponding monoids in the consideration. This is the corresponding derived differential complex attached to the canonical resolution (which we will denote it by the same notation as in the non logarithmic setting) of (B, N) :

$$(A, M)[(B, N)]^{\text{degreenumber}}, \quad (2.2.1)$$

which is now denoted by $\text{Kan}_{\text{Left}}\text{deRham}_{(B,N)/(A,M),\text{alg}}^{\text{degreenumber}} := \text{Kan}_{\text{Left}}\text{deRham}_{-(A,M),\text{alg}}^{\text{degreenumber}}((B, N))$ after taking the left Kan extension as in [B1, Chapter 6] and applying to the ring (B, N) . The corresponding cotangent complex associated is defined to be just:

$$\mathbb{L}_{(B,N)/(A,M),\text{alg}} := \text{Kan}_{\text{Left}}\text{deRham}_{(A,M)[(B,N)]^{\text{degreenumber}}/(A,M),\text{alg}}^1 \otimes_{(A,M)[(B,N)]^{\text{degreenumber}}} (B, N). \quad (2.2.2)$$

The corresponding algebraic André-Quillen homologies are defined to be:

$$H_{\text{degreenumber},\text{AQ},\text{alg}} := \pi_{\text{degreenumber}}(\mathbb{L}_{(B,N)/(A,M),\text{alg}}). \quad (2.2.3)$$

The corresponding topological André-Quillen complex is actually the complete version of the corresponding algebraic ones above by considering the corresponding certain p -completion over the simplicial module structure.

Then we consider the corresponding derived algebraic de Rham complex which is just defined to be:

$$\text{Kan}_{\text{Left}}\text{deRham}_{(B,N)/(A,M),\text{alg}}^{\text{degreenumber}}, \text{Fil}_{\text{Kan}_{\text{Left}}\text{deRham}_{(B,N)/(A,M),\text{alg}}^1}^*. \quad (2.2.4)$$

We then take the corresponding Banach completion and we denote that by:

$$\text{Kan}_{\text{Left}}\text{deRham}_{(B,N)/(A,M),\text{topo}}^{\text{degreenumber}}, \text{Fil}_{\text{Kan}_{\text{Left}}\text{deRham}_{(B,N)/(A,M),\text{topo}}^1}^*. \quad (2.2.5)$$

Then we need to take the corresponding Hodge-Filtered completion by using the corresponding filtration associated as above:

$$\text{Kan}_{\text{Left}}\widehat{\text{deRham}}_{(B,N)/(A,M),\text{topo}}^{\text{degreenumber}}, \text{Fil}_{\text{Kan}_{\text{Left}}\widehat{\text{deRham}}_{(B,N)/(A,M),\text{topo}}^1}^*. \quad (2.2.6)$$

This is basically the corresponding analytic and complete version the corresponding algebraic log de Rham complex. Furthermore we allow large coefficients with rigid affinoid algebra Z over \mathbb{Q}_p . Therefore we take the corresponding completed tensor product in the following.

Definition 2.2.1. We define the following Z deformed version of the corresponding complete version of the corresponding logarithmic André-Quillen homology and the corresponding complete version of the corresponding logarithmic derived de Rham complex. We start from the corresponding construction of the algebraic p -adic logarithmic derived de Rham complex for a map $(A, M) \rightarrow (B, N)$. Fix a pair of ring of definitions $(A_0, M_0), (B_0, N_0)$ in $(A, M), (B, N)$ respectively. Then this is the corresponding derived differential complex attached to the canonical resolution of (B_0, N_0) :

$$(A_0, M_0)[(B_0, N_0)]^{\text{degreenumber}}, \quad (2.2.7)$$

which is now denoted by $\text{Kan}_{\text{Left}}\text{deRham}_{(A_0, M_0)[(B_0, N_0)]^{\text{degreenumber}} / (A_0, M_0), \text{alg}}^{\text{degreenumber}}$ after taking the corresponding left Kan extension as in [B1, Chapter 6]. The corresponding cotangent complex associated is defined to be just:

$$\mathbb{L}_{B_0 / (A_0, M_0), \text{alg}} := \text{Kan}_{\text{Left}}\text{deRham}_{(A_0, M_0)[B_0, N_0]^{\text{degreenumber}} / (A_0, M_0), \text{alg}}^1 \otimes_{(A_0, M_0)[B_0, N_0]^{\text{degreenumber}}} (B_0, N_0). \quad (2.2.8)$$

The corresponding algebraic logarithmic André-Quillen homologies are defined to be:

$$H_{\text{degreenumber}, \text{AQ}, \text{alg}, \text{log}} := \pi_{\text{degreenumber}}(\mathbb{L}_{(B_0, N_0) / (A_0, M_0), \text{alg}}). \quad (2.2.9)$$

The corresponding topological logarithmic André-Quillen complex is actually the complete version of the corresponding algebraic ones above by considering the corresponding derived p -completion over the simplicial module structure:

$$\mathbb{L}_{(B_0, N_0) / (A_0, M_0), \text{topo}} \quad (2.2.10)$$

$$:= R \lim_{\leftarrow k} \text{Kos}_{p^k} \left(\left(\text{Kan}_{\text{Left}}\text{deRham}_{(A_0, M_0)[(B_0, N_0)]^{\text{degreenumber}} / (A_0, M_0), \text{alg}}^1 \otimes_{(A_0, N_0)[(B_0, N_0)]^{\text{degreenumber}}} (B_0, N_0) \right) \right). \quad (2.2.11)$$

Taking the product with \mathcal{O}_Z we have the corresponding integral version of the topological logarithmic André-Quillen complex:

$$\mathbb{L}_{(B_0, N_0) / (A_0, M_0), \text{topo}, Z} := \mathbb{L}_{(B_0, N_0) / (A_0, M_0), \text{topo}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_Z \quad (2.2.12)$$

Then we consider the corresponding derived algebraic de Rham complex which is just defined to be:

$$\text{Kan}_{\text{Left}}\text{deRham}_{(B_0, N_0) / (A_0, M_0), \text{alg}}^{\text{degreenumber}}, \text{Fil}_{\text{Kan}_{\text{Left}}\text{deRham}_{(B_0, N_0) / (A_0, M_0), \text{alg}}^1}^*. \quad (2.2.13)$$

We then take the corresponding derived p -completion and we denote that by:

$$\text{Kan}_{\text{Left}}\text{deRham}_{(B_0, N_0) / (A_0, M_0), \text{topo}}^{\text{degreenumber}} \quad (2.2.14)$$

$$:= R \lim_{\leftarrow k} \text{Kos}_{p^k} \left(\text{Kan}_{\text{Left}}\text{deRham}_{(B_0, N_0) / (A_0, M_0), \text{alg}}^{\text{degreenumber}}, \text{Fil}_{\text{Kan}_{\text{Left}}\text{deRham}_{(B_0, N_0) / (A_0, M_0), \text{alg}}^{\text{degreenumber}}}^* \right), \quad (2.2.15)$$

$$\text{Fil}_{\text{Kan}_{\text{Left}}\text{deRham}_{(B_0, N_0) / (A_0, M_0), \text{topo}}^{\text{degreenumber}}}^* := R \lim_{\leftarrow k} \text{Kos}_{p^k} \left(\text{Fil}_{\text{Kan}_{\text{Left}}\text{deRham}_{(B_0, N_0) / (A_0, M_0), \text{alg}}^{\text{degreenumber}}}^* \right). \quad (2.2.16)$$

Before considering the corresponding integral version we just consider the corresponding product of these \mathbb{E}_∞ -rings with \mathcal{O}_Z to get:

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}, Z}^{\mathrm{degreenumber}} := \mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}}^{\mathrm{degreenumber}} \widehat{\otimes}_{\mathbb{Z}_p}^{\mathbb{L}} \mathcal{O}_Z, \quad (2.2.17)$$

$$\mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}, Z}^{\mathrm{degreenumber}}}^* := \mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}}^{\mathrm{degreenumber}}}^* \widehat{\otimes}_{\mathbb{Z}_p}^{\mathbb{L}} \mathcal{O}_Z. \quad (2.2.18)$$

Then we consider the following construction for the map $(A, M) \rightarrow (B, N)$ by putting:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}, Z} := \mathrm{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} \mathbb{L}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}, Z}[1/p], \quad (2.2.19)$$

$$H_{\mathrm{degreenumber}, \mathrm{AQ}, \mathrm{topo}, Z} := \pi_{\mathrm{degreenumber}}(\mathbb{L}_{(B, N)/(A, M), \mathrm{topo}, Z}), \quad (2.2.20)$$

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B, N)/(A, M), \mathrm{topo}, Z}^{\mathrm{degreenumber}} := \mathrm{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} \mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}, Z}^{\mathrm{degreenumber}}[1/p], \quad (2.2.21)$$

$$\mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B, N)/(A, M), \mathrm{topo}, Z}^{\mathrm{degreenumber}}}^* := \mathrm{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} \mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}, Z}^{\mathrm{degreenumber}}}^*[1/p]. \quad (2.2.22)$$

Then we need to take the corresponding Hodge-Filtered completion by using the corresponding filtration associated as above to achieve the corresponding Hodge-complete objects in the corresponding filtered ∞ -categories:

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B, N)/(A, M), \mathrm{topo}, \mathrm{Hodge}, Z}^{\mathrm{degreenumber}} \mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B, N)/(A, M), \mathrm{topo}, \mathrm{Hodge}, Z}^1}^*. \quad (2.2.23)$$

Definition 2.2.2. We define the corresponding finite projective filtered crystals to be the corresponding finite projective module spectra over the topological filtered E_∞ -ring $\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B, N)/(A, M), \mathrm{topo}, Z}^{\mathrm{degreenumber}}$ with the corresponding induced filtrations.

Definition 2.2.3. We define the corresponding almost perfect¹⁵ filtered crystals to be the corresponding almost perfect module spectra over the topological filtered E_∞ -ring $\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B, N)/(A, N), \mathrm{topo}, Z}^{\mathrm{degreenumber}}$ with the corresponding induced filtrations.

The following is inspired by the main Poincaré Lemma from [GL, Theorem 1.2] in the non-deformed situation. Consider a corresponding smooth log rigid analytic space X over k/\mathbb{Q}_p (where k is a corresponding unramified analytic field which is discretely-valued and the corresponding residue field is finite). Then we have the following:

Conjecture 2.2.4. Consider the corresponding projective map $g : X_{\mathrm{Kummer-pro-étale}} \rightarrow X_{\mathrm{Kummer-ét}}$ and the the projective map $f : X_{\mathrm{Kummer-pro-étale}} \rightarrow X$. Then we have the logarithmic versions of the strictly exact long exact sequences as in the Poincaré lemma in the rigid analytic situation.

¹⁵This is the corresponding derived version of pseudocoherence from [Lu1], [Lu2].

2.2.2 Logarithmic Setting for Pseudorigid Spaces

We now consider the analytic logarithmic André-Quillen Homology and analytic logarithmic Derived de Rham complex of pseudorigid space, which are very crucial in some development in [Bel1] in the arithmetic family. Therefore we just investigate the corresponding picture in the corresponding geometric family.

Setting 2.2.5. We consider now a corresponding morphism taking the corresponding form of $(A, M) \rightarrow (B, N)$ where (A, M) is going to be a log pseudorigid affinoid algebra over \mathbb{Z}_p (where we consider the corresponding foundation in [DLLZ1, Chapter 2] for the corresponding log adic rings), and (B, N) is going to be a Kummer perfectoid chart of A in the corresponding Kummer pro-étale site of the log pseudorigid affinoid space attached to A (where we consider the corresponding foundation in [DLLZ1, Chapter 5] for the corresponding Kummer pro-étale sites). As in [Bel1, Definition 3.1, and below Definition 3.1] in our situation A is of topologically finite type over $\mathcal{O}_K[[t]] \langle \pi^a / t^b \rangle [1/t]$, where K is a discrete valued field containing \mathbb{Q}_p and $(a, b) = 1$ ¹⁶.

From [Bel1, Definition 3.1, and below Definition 3.1] we have the following:

Lemma 2.2.6. *We have the following statements:*

- A. *A as above is Tate, complete over \mathcal{O}_K ;*
- B. *The ring A has a ring of definition A_0 which is of \mathcal{O}_K -formally finite type;*
- C. *The ring A_0 is of topologically finite type over $\mathcal{O}_K[[t]] \langle \pi^a / t^b \rangle$.*

Proof. Since we did not change the corresponding assumption on the corresponding ring theoretic consideration. \square

As in the corresponding construction in the rigid situation in the previous section following [An1, Chapitre 3], [An2], [B1, Chapter 5, Chapter 6, Chapter 7], [Bei, Chapter 1], [G1, Chapter 5], [GL, Chapter 3, Chapter 4], [Ill1, Chapitre II, Chapitre III], [Ill2, Chapitre VIII], [O, Chapter 8], [Qui, Section 4]. We keep now the following setting:

Setting 2.2.7. Now as in the above notion we will consider a general map of logarithmic rings $(A, M) \rightarrow (B, N)$ over $\mathcal{O}_K[[t]] \langle \pi^a / t^b \rangle [1/t]$ where we have the corresponding map of the associated ring of definitions $(A_0, M_0) \rightarrow (B_0, N_0)$ over $\mathcal{O}_K[[t]] \langle \pi^a / t^b \rangle$ such that A_0, B_0 are basically I -adic (where $\mathcal{O}_K[[t]] \langle \pi^a / t^b \rangle$ is I -adic).

Definition 2.2.8. We start from the corresponding construction of the algebraic p -adic log derived de Rham complex for a map $(A, M) \rightarrow (B, N)$. Fix a pair of ring of definitions $(A_0, M_0), (B_0, N_0)$ in $(A, M), (B, N)$ respectively. Then this is the corresponding derived differential complex attached to the canonical resolution of (B_0, N_0) :

$$(A_0, M_0)[(B_0, N_0)]^{\text{degreenumber}}, \quad (2.2.24)$$

which is now denoted by $\text{Kan}_{\text{Left}} \text{deRham}_{(B_0, N_0)/(A_0, M_0), \text{alg}}^{\text{degreenumber}}$. This is after taking the left Kan extension as in [B1, Chapter 6]. The corresponding logarithmic cotangent complex associated is defined to be just:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{alg}} := \text{Kan}_{\text{Left}} \text{deRham}_{(A_0, M_0)[(B_0, N_0)]^{\text{degreenumber}} / (A_0, M_0), \text{alg}}^1 \otimes_{(A_0, M_0)[(B_0, N_0)]^{\text{degreenumber}}} (B_0, N_0). \quad (2.2.25)$$

¹⁶Certainly one can also consider the characteristic p situation.

The corresponding algebraic logarithmic André-Quillen homologies are defined to be:

$$H_{\text{degreenumber}, \text{AQ}, \text{alg}} := \pi_{\text{degreenumber}}(\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{alg}}). \quad (2.2.26)$$

The corresponding topological logarithmic André-Quillen complex is actually the complete version of the corresponding algebraic ones above by considering the corresponding derived I -completion over the simplicial module structure:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}} \quad (2.2.27)$$

$$:= R \lim_{\leftarrow I} \text{Kos}_I \left((\text{Kan}_{\text{Left}} \text{deRham}_{(A_0, M_0)[(B_0, N_0)]^{\text{degreenumber}} / (A_0, M_0), \text{alg}}^1 \otimes_{(A_0, M_0)[(B_0, N_0)]^{\text{degreenumber}}} (B_0, N_0)) \right). \quad (2.2.28)$$

Then we consider the corresponding logarithmic derived algebraic de Rham complex which is just defined to be:

$$\text{Kan}_{\text{Left}} \text{deRham}_{(B_0, N_0)/(A_0, M_0), \text{alg}}^{\text{degreenumber}}, \text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{(B_0, N_0)/(A_0, M_0), \text{alg}}}^*. \quad (2.2.29)$$

We then take the corresponding derived I -completion and we denote that by:

$$\text{Kan}_{\text{Left}} \text{deRham}_{(B_0, N_0)/(A_0, M_0), \text{topo}}^{\text{degreenumber}} := \quad (2.2.30)$$

$$R \lim_{\leftarrow I} \text{Kos}_I \left(\text{Kan}_{\text{Left}} \text{deRham}_{(A_0, M_0)[(B_0, N_0)]^{\text{degreenumber}} / (A_0, M_0), \text{alg}}^{\text{degreenumber}}, \text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{(B_0, N_0)/(A_0, M_0), \text{alg}}}^{\text{degreenumber}} \right), \quad (2.2.31)$$

$$\text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{(B_0, N_0)/(A_0, M_0), \text{topo}}^{\text{degreenumber}}}^* := R \lim_{\leftarrow I} \text{Kos}_I \left(\text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{(B_0, N_0)/(A_0, M_0), \text{alg}}}^{\text{degreenumber}} \right). \quad (2.2.32)$$

Then we consider the following construction for the map $(A, M) \rightarrow (B, N)$ by putting:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}} := \text{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} \mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}}[1/t], \quad (2.2.33)$$

$$H_{\text{degreenumber}, \text{AQ}, \text{topo}} := \pi_{\text{degreenumber}}(\mathbb{L}_{(B, M)/(A, M), \text{topo}}), \quad (2.2.34)$$

$$\text{Kan}_{\text{Left}} \text{deRham}_{(B, N)/(A, M), \text{topo}}^{\text{degreenumber}} := \text{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} \text{Kan}_{\text{Left}} \text{deRham}_{(B_0, N_0)/(A_0, M_0), \text{topo}}^{\text{degreenumber}}[1/t], \quad (2.2.35)$$

$$\text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{(B, N)/(A, M), \text{topo}}^{\text{degreenumber}}}^* := \text{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} \text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{(B_0, N_0)/(A_0, M_0), \text{topo}}^{\text{degreenumber}}}^*[1/t]. \quad (2.2.36)$$

Then we need to take the corresponding Hodge-Filtered completion by using the corresponding filtration associated as above to achieve the corresponding Hodge-complete objects in the corresponding filtered ∞ -categories:

$$\text{Kan}_{\text{Left}} \text{deRham}_{(B, N)/(A, M), \text{topo}, \text{Hodge}}^{\text{degreenumber}}, \text{Fil}_{\text{Kan}_{\text{Left}} \text{deRham}_{(B, N)/(A, M), \text{topo}, \text{Hodge}}^{\text{degreenumber}}}^*. \quad (2.2.37)$$

Definition 2.2.9. We define the corresponding finite projective filtered crystals to be the corresponding finite projective module spectra over the topological filtered E_∞ -ring $\text{Kan}_{\text{Left}} \text{deRham}_{(B, N)/(A, M), \text{topo}}^{\text{degreenumber}}$ with the corresponding induced filtrations.

Definition 2.2.10. We define the corresponding almost perfect ¹⁷ filtered crystals to be the corresponding almost perfect module spectra over the topological filtered E_∞ -ring $\text{Kan}_{\text{Left}}\text{deRham}_{(B,N)/(A,M),\text{topo}}^{\text{degree number}}$ with the corresponding induced filtrations.

We now consider the corresponding large coefficient local systems over logarithmic pseudorigid spaces where Z now is a topological algebra over \mathbb{Z}_p ¹⁸.

Definition 2.2.11. We define the following Z deformed version of the corresponding complete version of the corresponding logarithmic André-Quillen homology and the corresponding complete version of the corresponding logarithmic derived de Rham complex. We start from the corresponding construction of the logarithmic algebraic p -adic derived de Rham complex for a map $(A, M) \rightarrow (B, N)$. Fix a pair of ring of definitions $(A_0, M_0), (B_0, N_0)$ in $(A, M), (B, N)$ respectively. Then this is the corresponding derived differential complex attached to the canonical resolution of (B_0, N_0) :

$$(A_0, M_0)[(B_0, N_0)]^{\text{degree number}}, \quad (2.2.38)$$

which is now denoted by $\text{Kan}_{\text{Left}}\text{deRham}_{(B_0, N_0)/(A_0, M_0), \text{alg}}^{\text{degree number}}$. The corresponding logarithmic cotangent complex associated is defined to be just:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{alg}} := \text{Kan}_{\text{Left}}\text{deRham}_{(A_0, M_0)[(B_0, N_0)]^{\text{degree number}} / (A_0, M_0), \text{alg}}^1 \otimes_{(A_0, M_0)[(B_0, N_0)]^{\text{degree number}}} (B_0, N_0). \quad (2.2.39)$$

The corresponding algebraic logarithmic André-Quillen homologies are defined to be:

$$H_{\text{degree number}, \text{AQ}, \text{alg}} := \pi_{\text{degree number}}(\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{alg}}). \quad (2.2.40)$$

The corresponding topological logarithmic André-Quillen complex is actually the complete version of the corresponding algebraic ones above by considering the corresponding derived I -completion over the simplicial module structure:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}} \quad (2.2.41)$$

$$:= R \varprojlim_k \text{Kos}_I \left((\text{Kan}_{\text{Left}}\text{deRham}_{(A_0, M_0)[(B_0, N_0)]^{\text{degree number}} / (A_0, M_0), \text{alg}}^1 \otimes_{(A_0, M_0)[(B_0, N_0)]^{\text{degree number}}} (B_0, N_0)) \right). \quad (2.2.42)$$

Taking the product with Z we have the corresponding integral version of the topological logarithmic version of the corresponding André-Quillen complex:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}, Z} := \mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}} \otimes_{\mathbb{Z}_p} \mathcal{O}_Z \quad (2.2.43)$$

Then we consider the corresponding logarithmic derived algebraic de Rham complex which is just defined to be:

$$\text{Kan}_{\text{Left}}\text{deRham}_{(B_0, N_0)/(A_0, M_0), \text{alg}}^{\text{degree number}}, \text{Fil}_{\text{Kan}_{\text{Left}}\text{deRham}_{(B_0, N_0)/(A_0, M_0), \text{alg}}^1}^*. \quad (2.2.44)$$

¹⁷This is the corresponding derived version of pseudocoherence from [Lu1], [Lu2].

¹⁸It is better to assume that it is basically completely flat.

We then take the corresponding derived I -completion and we denote that by:

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}}^{\mathrm{degreenumber}} := \quad (2.2.45)$$

$$R \lim_{\leftarrow k} \mathrm{Kos}_I \left(\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{alg}}^{\mathrm{degreenumber}}, \mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{alg}}^{\mathrm{degreenumber}}}^* \right), \quad (2.2.46)$$

$$\mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}}^{\mathrm{degreenumber}}}^* := R \lim_{\leftarrow k} \mathrm{Kos}_I \left(\mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{alg}}^{\mathrm{degreenumber}}}^* \right). \quad (2.2.47)$$

Before considering the corresponding integral version we just consider the corresponding product of these \mathbb{E}_∞ -rings with Z to get:

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}, Z}^{\mathrm{degreenumber}} := \mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}}^{\mathrm{degreenumber}} \otimes_{\mathbb{Z}_p}^{\mathbb{L}} Z, \quad (2.2.48)$$

$$\mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}, Z}^{\mathrm{degreenumber}}}^* := \mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}}^{\mathrm{degreenumber}}}^* \otimes_{\mathbb{Z}_p}^{\mathbb{L}} Z. \quad (2.2.49)$$

Then we consider the following construction for the map $(A, M) \rightarrow (B, N)$ by putting:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}, Z} := \mathrm{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} \mathbb{L}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}, Z}[1/t], \quad (2.2.50)$$

$$H_{\mathrm{degreenumber}, \mathrm{AQ}, \mathrm{topo}, Z} := \pi_{\mathrm{degreenumber}}(\mathbb{L}_{(B, N)/(A, M), \mathrm{topo}, Z}), \quad (2.2.51)$$

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B, N)/(A, M), \mathrm{topo}, Z}^{\mathrm{degreenumber}} := \mathrm{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} \mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}, Z}^{\mathrm{degreenumber}}[1/t], \quad (2.2.52)$$

$$\mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B, N)/(A, M), \mathrm{topo}, Z}^{\mathrm{degreenumber}}}^* := \mathrm{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} \mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}, Z}^{\mathrm{degreenumber}}}^*[1/t]. \quad (2.2.53)$$

Then we need to take the corresponding Hodge-Filtered completion by using the corresponding filtration associated as above to achieve the corresponding Hodge-complete objects in the corresponding filtered ∞ -categories:

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B, N)/(A, M), \mathrm{topo}, \mathrm{Hodge}, Z}^{\mathrm{degreenumber}}, \mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B, N)/(A, M), \mathrm{topo}, \mathrm{Hodge}, Z}^1}^*. \quad (2.2.54)$$

Example 2.2.12. Now we construct the corresponding pseudorigid analog of the corresponding example in [GL, Example 4.7]. Now we consider the corresponding the map:

$$\mathbb{Z}_p[[u]] \left\langle \frac{p^a}{u^b} \right\rangle [1/u] \langle X_1^\pm, X_2^\pm, \dots, X_d^\pm, X_{d+1}, \dots, X_e \rangle \quad (2.2.55)$$

$$\longrightarrow \mathbb{Z}_p[[u]] \left\langle \frac{p^a}{u^b} \right\rangle [1/u] \langle X_1^{\pm/p^\infty}, X_2^{\pm/p^\infty}, \dots, X_d^{\pm/p^\infty}, X_{d+1}^{1/p^\infty}, \dots, X_e^{1/p^\infty} \rangle \quad (2.2.56)$$

which could be written as:

$$\mathbb{Z}_p[[u]] \left\langle \frac{p^a}{u^b} \right\rangle [1/u] \langle X_1^\pm, X_2^\pm, \dots, X_d^\pm, X_{d+1}, \dots, X_e \rangle \longrightarrow \quad (2.2.57)$$

$$\mathbb{Z}_p[[u]] \left\langle \frac{p^a}{u^b} \right\rangle [1/u] \langle X_1^\pm, X_2^\pm, \dots, X_d^\pm, X_{d+1}, \dots, X_e \rangle \langle Y_1^{\pm/p^\infty}, \dots, Y_d^{\pm/p^\infty}, Y_{d+1}^{1/p^\infty}, \dots, Y_e^{1/p^\infty} \rangle \quad (2.2.58)$$

$$/(X_i - Y_i, i = 1, \dots, e). \quad (2.2.59)$$

So in our situation the corresponding Hodge complete topological derived de Rham complex will be just:

$$\mathbb{Z}_p[[u]] \left\langle \frac{p^a}{u^b} \right\rangle [1/u] \left\langle Y_1^{\pm/p^\infty}, Y_2^{\pm/p^\infty}, \dots, Y_d^{\pm/p^\infty}, Y_{d+1}^{1/p^\infty}, \dots, Y_e^{1/p^\infty} \right\rangle \quad (2.2.60)$$

$$[[Z_1, \dots, Z_e, Z_i = X_i - Y_i, i = 1, \dots, e]]. \quad (2.2.61)$$

The following conjectures are literally inspired by the rigid analytic situation in [GL, Theorem 1.2]. We assume the spaces are smooth.

Conjecture 2.2.13. *After the whole foundation of [DLLZ1] and [Ked1, Theorem 2.9.9, Remark 2.9.10], the construction in this section could be carried over the corresponding certain Kummer-pro-étale sites where logarithmic perfectoid subdomains form a corresponding basis of neighbourhood of the topology.*

Based on this conjecture one can conjecture the following:

Conjecture 2.2.14. *Consider the corresponding projective map $g : X_{\text{Kummer-pro-étale}} \rightarrow X_{\text{Kummer-ét}}$ and the the projective map $f : X_{\text{Kummer-pro-étale}} \rightarrow X$. Then we have the log versions of the strictly exact Poincaré long exact sequences as in the pseudorigid analytic situation.*

and

Conjecture 2.2.15. *Consider the corresponding projective map $g : X_{\text{Kummer-pro-étale}} \rightarrow X_{\text{Kummer-ét}}$ and the the projective map $f : X_{\text{Kummer-pro-étale}} \rightarrow X$. Let M be a corresponding Z -projective differential crystal spectrum. Then we have the logarithmic versions of the strictly exact Poincaré long exact sequences for M as in the pseudorigid analytic situation.*

2.2.3 Derived (p, I) -Complete Logarithmic Derived de Rham complex

The corresponding construction of the derived de Rham complex could be defined for general derived spaces. Along our discussion in the situations of rigid analytic spaces and the corresponding pseudorigid spaces we now focus on the corresponding derived rings. And we consider the corresponding logarithmic setting.

Setting 2.2.16. We now fix a bounded morphism of log simplicial topological rings $(A, M) \rightarrow (B, N)$ over A^* where A^* contains a corresponding ring of definition A_0^* which is derived complete with respect to the (p, I) -topology and we assume that (A, M) is adic in the sense to be defined just below and we assume that (A_0^*, I) is a prism namely we at least require that the corresponding δ -structure on the corresponding ring will induce the map $\varphi(\cdot) := \cdot^p + p\delta(\cdot)$ such that we have the situation where $p \in (I, \varphi(I))$. For (A, M) or (B, N) respectively we assume this contains a subring (A_0, M) or (B_0, N_0) (over A_0^*) respectively such that we have A_0 or B_0 respectively is derived complete with respect to the corresponding derived (p, I) -topology and we assume that $B = B_0[1/f, f \in I]$ (same for A)¹⁹. All the adic rings are assumed to be open mapping. We use the notation d to denote a corresponding primitive element as in [BS1, Section 2.3] for A^* . We are going to assume that p is a topologically nilpotent element.

As in the corresponding non-logarithmic situation following [B1, Chapter 6]²⁰ we first consider the category $\text{Alg}_{\mathfrak{G}_{\text{prelog, smooth}, A_0^*}}$ of all the smooth prelog algebras over A_0^* and take the corresponding left Kan extension to the corresponding ∞ -category $\text{Alg}_{\mathfrak{G}_{\infty, \text{prelog}, A_0^*}}$ of all the prelog simplicial A_0^* -algebras. Then we may take the corresponding derived (p, I) -completion to get completed objects. Finally we could basically apply this to the relatively more specific rings in the previous setting. These are basically parallel to the situations where we work over \mathbb{Z}_p and the pseudorigid disc²¹.

Remark 2.2.17. The corresponding idea behind this is certainly inspired by [LL] for instance where the authors compared the corresponding prismatic cohomology of some suitable ring R over A_0^*/I for instance and the corresponding derived de Rham cohomology canonically attached to the corresponding morphism, and the corresponding Hodge filtration is compared to the Nygaard filtration.

As in the corresponding construction in the rigid and pseudocoherent situation in the previous section following [An1, Chapitre 3], [An2], [B1, Chapter 5, Chapter 6, Chapter 7], [Bei, Chapter 1], [G1, Chapter 5], [GL, Chapter 3, Chapter 4], [Ill1, Chapitre II, Chapitre III], [Ill2, Chapitre VIII], [O, Chapter 8], [Qui, Section 4] we give the following definition.

Definition 2.2.18. We start from the corresponding construction of the algebraic p -adic logarithmic derived de Rham complex for a map $(A, M) \rightarrow (B, N)$. Fix a pair of ring of definitions $(A_0, M_0), (B_0, N_0)$ in $(A, M), (B, N)$ respectively. Then this is the corresponding derived differential complex attached to the cofibrant replacement of (B_0, N_0) :

$$P_{(B_0, N_0)}^{\text{degree number}}, \quad (2.2.62)$$

¹⁹One can also invert p to consider the rationality with respect to p as in the rigid analytic situation.

²⁰Here the corresponding left Kan extension happens along the embedding of the free prelog rings to the ∞ -category of simplicial prelog rings. See [B1, Chapter 6] for the construction.

²¹Certainly we could work in more general setting over any derived J -complete ring, such as in the situation where we do not require that the ring A_0^* is a prism.

which is now denoted by $\text{Kan}_{\text{Left}}\text{deRham}_{(B_0, N_0)/(A_0, M_0), \text{alg}}^{\text{degreenumber}}$ after taking the corresponding left Kan extension as in [B1, Chapter 6]. The corresponding logarithmic cotangent complex associated is defined to be just:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{alg}} := \text{Kan}_{\text{Left}}\text{deRham}_{P_{(B_0, N_0)}^{\text{degreenumber}}/(A_0, M_0), \text{alg}}^1 \otimes_{P_{(B_0, N_0)}^{\text{degreenumber}}} (B_0, N_0). \quad (2.2.63)$$

The corresponding logarithmic algebraic André-Quillen homologies are defined to be:

$$H_{\text{degreenumber}, \text{AQ}, \text{alg}} := \pi_{\text{degreenumber}}(\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{alg}}). \quad (2.2.64)$$

The corresponding topological André-Quillen complex is actually the complete version of the corresponding algebraic ones above by considering the corresponding derived (p, I) -completion over the simplicial module structure:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}} := R \varprojlim_{(p, I)} \text{Kos}_{(p, I)} \left(\left(\text{Kan}_{\text{Left}}\text{deRham}_{P_{(B_0, N_0)}^{\text{degreenumber}}/(A_0, M_0), \text{alg}}^1 \otimes_{P_{(B_0, N_0)}^{\text{degreenumber}}} (B_0, N_0) \right) \right). \quad (2.2.65)$$

Then we consider the corresponding derived algebraic de Rham complex which is just defined to be:

$$\text{Kan}_{\text{Left}}\text{deRham}_{(B_0, N_0)/(A_0, M_0), \text{alg}}^{\text{degreenumber}} \text{Fil}_{\text{Kan}_{\text{Left}}\text{deRham}_{(B_0, N_0)/(A_0, M_0), \text{alg}}^1}^*. \quad (2.2.66)$$

We then take the corresponding derived (p, I) -completion and we denote that by:

$$\text{Kan}_{\text{Left}}\text{deRham}_{(B_0, N_0)/(A_0, M_0), \text{topo}}^{\text{degreenumber}} \quad (2.2.67)$$

$$:= R \varprojlim_{(p, I)} \text{Kos}_{(p, I)} \left(\text{Kan}_{\text{Left}}\text{deRham}_{(B_0, N_0)/(A_0, M_0), \text{alg}}^{\text{degreenumber}} \text{Fil}_{\text{Kan}_{\text{Left}}\text{deRham}_{(B_0, N_0)/(A_0, M_0), \text{alg}}^{\text{degreenumber}}}^* \right), \quad (2.2.68)$$

$$\text{Fil}_{\text{Kan}_{\text{Left}}\text{deRham}_{(B_0, N_0)/(A_0, M_0), \text{topo}}^{\text{degreenumber}}}^* := R \varprojlim_{(p, I)} \text{Kos}_{(p, I)} \left(\text{Fil}_{\text{Kan}_{\text{Left}}\text{deRham}_{(B_0, N_0)/(A_0, M_0), \text{alg}}^{\text{degreenumber}}}^* \right). \quad (2.2.69)$$

Then in the situation that all the rings are classical logarithmic adic rings we consider the following construction for the map $A \rightarrow B$ by putting:

$$\mathbb{L}_{(B, N)/(A, M), \text{topo}} := \text{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} \mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}}[1/(d)], \quad (2.2.70)$$

$$H_{\text{degreenumber}, \text{AQ}, \text{topo}} := \pi_{\text{degreenumber}}(\mathbb{L}_{(B, N)/(A, M), \text{topo}}), \quad (2.2.71)$$

$$\text{Kan}_{\text{Left}}\text{deRham}_{(B, N)/(A, M), \text{topo}}^{\text{degreenumber}} := \text{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} \text{Kan}_{\text{Left}}\text{deRham}_{(B_0, N_0)/(A_0, M_0), \text{topo}}^{\text{degreenumber}}[1/(d)], \quad (2.2.72)$$

$$\text{Fil}_{\text{Kan}_{\text{Left}}\text{deRham}_{(B, N)/(A, M), \text{topo}}^{\text{degreenumber}}}^* := \text{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} \text{Fil}_{\text{Kan}_{\text{Left}}\text{deRham}_{(B_0, N_0)/(A_0, M_0), \text{topo}}^{\text{degreenumber}}}^*[1/(d)]. \quad (2.2.73)$$

Then we need to take the corresponding Hodge-Filtered completion by using the corresponding filtration associated as above to achieve the corresponding Hodge-complete objects in the corresponding filtered ∞ -categories:

$$\text{Kan}_{\text{Left}}\text{deRham}_{(B, N)/(A, M), \text{topo}, \text{Hodge}}^{\text{degreenumber}} \text{Fil}_{\text{Kan}_{\text{Left}}\text{deRham}_{(B, N)/(A, M), \text{topo}, \text{Hodge}}^{\text{degreenumber}}}^*. \quad (2.2.74)$$

Definition 2.2.19. We define the corresponding finite projective filtered crystals to be the corresponding finite projective module spectra over the topological filtered E_∞ -ring $\text{Kan}_{\text{Left}}\text{deRham}_{(B,N)/(A,M),\text{topo}}^{\text{degreenumber}}$ with the corresponding induced filtrations.

Definition 2.2.20. We define the corresponding almost perfect²² filtered crystals to be the corresponding almost perfect module spectra over the topological filtered E_∞ -ring $\text{Kan}_{\text{Left}}\text{deRham}_{(B,N)/(A,M),\text{topo}}^{\text{degreenumber}}$ with the corresponding induced filtrations.

We now consider the corresponding large coefficient local systems over pseudorigid spaces where Z now is a simplicial topological algebra over \mathbb{Z}_p ²³.

Definition 2.2.21. We define the following Z deformed version of the corresponding complete version of the corresponding logarithmic André-Quillen homology and the corresponding complete version of the corresponding logarithmic derived de Rham complex. We start from the corresponding construction of the logarithmic algebraic derived de Rham complex for a map $(A, M) \rightarrow (B, N)$. Fix a pair of ring of definitions $(A_0, M_0), (B_0, N_0)$ in $(A, M), (B, N)$ respectively. Then this is the corresponding derived differential complex attached to the canonical resolution of (B_0, N_0) :

$$(A_0, M_0)[(B_0, N_0)]^{\text{degreenumber}}, \quad (2.2.75)$$

which is now denoted by $\text{Kan}_{\text{Left}}\text{deRham}_{(B_0, N_0)/(A_0, M_0), \text{alg}}^{\text{degreenumber}}$. The corresponding logarithmic cotangent complex associated is defined to be just:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{alg}} \quad (2.2.76)$$

$$:= \text{Kan}_{\text{Left}}\text{deRham}_{(A_0, M_0)[(B_0, N_0)]^{\text{degreenumber}} / (A_0, M_0), \text{alg}}^1 \otimes_{(A_0, M_0)[(B_0, N_0)]^{\text{degreenumber}}} (B_0, N_0). \quad (2.2.77)$$

The corresponding algebraic logarithmic André-Quillen homologies are defined to be:

$$H_{\text{degreenumber}, \text{AQ}, \text{alg}} := \pi_{\text{degreenumber}}(\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{alg}}). \quad (2.2.78)$$

The corresponding topological logarithmic André-Quillen complex is actually the complete version of the corresponding algebraic ones above by considering the corresponding derived (p, I) -completion over the simplicial module structure:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}} \quad (2.2.79)$$

$$:= R \varprojlim_k \text{Kos}_{(p, I)} \left(\left(\text{Kan}_{\text{Left}}\text{deRham}_{(A_0, M_0)[(B_0, N_0)]^{\text{degreenumber}} / (A_0, M_0), \text{alg}}^1 \otimes_{(A_0, M_0)[(B_0, N_0)]^{\text{degreenumber}}} (B_0, N_0) \right) \right). \quad (2.2.80)$$

Taking the product with Z we have the corresponding integral version of the topological logarithmic version of the corresponding André-Quillen complex:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}, Z} := \mathbb{L}_{(B_0, N_0)/(A_0, M_0), \text{topo}} \otimes_{\mathbb{Z}_p} Z \quad (2.2.81)$$

²²This is the corresponding derived version of pseudocoherence from [Lu1], [Lu2].

²³It is better to assume that it is basically derived completely flat.

Then we consider the corresponding logarithmic derived algebraic de Rham complex which is just defined to be:

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{alg}}^{\mathrm{degreenumber}}, \mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{alg}}^1}^* \quad (2.2.82)$$

We then take the corresponding derived (p, I) -completion and we denote that by:

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}}^{\mathrm{degreenumber}} := \quad (2.2.83)$$

$$R \lim_{\leftarrow k} \mathrm{Kos}_{(p, I)} \left(\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{alg}}^{\mathrm{degreenumber}}, \mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{alg}}^{\mathrm{degreenumber}}}^* \right), \quad (2.2.84)$$

$$\mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}}^{\mathrm{degreenumber}}}^* := R \lim_{\leftarrow k} \mathrm{Kos}_{(p, I)} \left(\mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{alg}}^{\mathrm{degreenumber}}}^* \right). \quad (2.2.85)$$

Before considering the corresponding integral version we just consider the corresponding product of these \mathbb{E}_∞ -rings with Z to get:

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}, Z}^{\mathrm{degreenumber}} := \mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}}^{\mathrm{degreenumber}} \otimes_{\mathbb{Z}_p}^{\mathbb{L}} Z, \quad (2.2.86)$$

$$\mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}, Z}^{\mathrm{degreenumber}}}^* := \mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}}^{\mathrm{degreenumber}}}^* \otimes_{\mathbb{Z}_p}^{\mathbb{L}} Z. \quad (2.2.87)$$

When we have that the corresponding ring Z is also derived (p, I) -topologized and commutative, then we can further take the corresponding derived (p, I) -completion to achieve the corresponding derived completed version:

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}, Z}^{\mathrm{degreenumber}} := \mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}}^{\mathrm{degreenumber}} \widehat{\otimes}_{\mathbb{Z}_p}^{\mathbb{L}} Z, \quad (2.2.88)$$

$$\mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}, Z}^{\mathrm{degreenumber}}}^* := \mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}}^{\mathrm{degreenumber}}}^* \widehat{\otimes}_{\mathbb{Z}_p}^{\mathbb{L}} Z. \quad (2.2.89)$$

Then in the situation that all the rings are classical logarithmic adic rings we consider the following construction for the map $(A, M) \rightarrow (B, N)$ by putting:

$$\mathbb{L}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}, Z} := \mathrm{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} \mathbb{L}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}, Z} [1/(p, I)], \quad (2.2.90)$$

$$H_{\mathrm{degreenumber}, \mathrm{AQ}, \mathrm{topo}, Z} := \pi_{\mathrm{degreenumber}}(\mathbb{L}_{(B, N)/(A, M), \mathrm{topo}, Z}), \quad (2.2.91)$$

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B, N)/(A, M), \mathrm{topo}, Z}^{\mathrm{degreenumber}} := \mathrm{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} \mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B, N)/(A_0, M_0), \mathrm{topo}, Z}^{\mathrm{degreenumber}} [1/(p, I)], \quad (2.2.92)$$

$$\mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B, N)/(A, M), \mathrm{topo}, Z}^{\mathrm{degreenumber}}}^* := \mathrm{Colim}_{(A_0, M_0) \rightarrow (B_0, N_0)} \mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B_0, N_0)/(A_0, M_0), \mathrm{topo}, Z}^{\mathrm{degreenumber}}}^* [1/(p, I)]. \quad (2.2.93)$$

Then we need to take the corresponding Hodge-Filtered completion by using the corresponding filtration associated as above to achieve the corresponding Hodge-complete objects in the corresponding filtered ∞ -categories:

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B, N)/(A, M), \mathrm{topo}, \mathrm{Hodge}, Z}^{\mathrm{degreenumber}}, \mathrm{Fil}_{\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{(B, N)/(A, M), \mathrm{topo}, \mathrm{Hodge}, Z}^{\mathrm{degreenumber}}}^* \quad (2.2.94)$$

2.3 Robba Sheaves and Frobenius Sheaves

2.3.1 Pseudorigid Relative Toric Tower

In this section we discuss the corresponding Robba sheaves of [KL1] and [KL2] over general spaces. Here we consider the corresponding discussion for the pseudorigid situation which is certainly following the corresponding treatment in [KL2, Chapter 8]. But we do not really understand if the corresponding generality could be achieved at this moment. So one definitely has to be very careful in the corresponding analysis ²⁴. These kinds of spaces are actually different from the rigid analytic space, although the theories are really related to each other, such as in [Bel1], [Bel2] and [L]. Following the ideas in the rigid situation in [KL2], we consider the towers in the smooth situation in the following:

Setting 2.3.1. We consider the following towers as in [KL2, Chapter 5]. First we let A_0 be the ring $\mathcal{O}_K[[u]] \left\langle \frac{\pi^a}{u^b} \right\rangle [1/u] \langle T_1^\pm, \dots, T_d^\pm \rangle$ and we put A_0^+ to be the ring $\mathcal{O}_K[[u]] \left\langle \frac{\pi^a}{u^b} \right\rangle \cdot \langle T_1^\pm, \dots, T_d^\pm \rangle$. And for the higher level we have the following rings:

$$(A_n, A_n^+) \tag{2.3.1}$$

$$:= (\mathcal{O}_K(\pi^{1/p^n})[[u]] \left\langle \frac{\pi^a}{u^b} \right\rangle [1/u](u^{\pm 1/p^n}) \langle T_1^{\pm 1/p^n}, \dots, T_d^{\pm 1/p^n} \rangle, \tag{2.3.2}$$

$$\mathcal{O}_K(\pi^{1/p^n})[[u]] \left\langle \frac{\pi^a}{u^b} \right\rangle [u^{1/p^n}] \langle T_1^{\pm 1/p^n}, \dots, T_d^{\pm 1/p^n} \rangle), \tag{2.3.3}$$

which implies that we have:

$$A_0 \longrightarrow A_1 \longrightarrow \dots \longrightarrow A_n \longrightarrow \dots, \forall n \geq 0.$$

Proposition 2.3.2. *The tower above is finite étale.*

Proof. Note that the corresponding pseudorigid space is actually formally of finite type over \mathcal{O}_K namely we have that the ring at the zeroth level is just

$$\mathcal{O}_K[[u, S_1, \dots, S_d]][1/u] \langle V_1^\pm, \dots, V_e^\pm \rangle.$$

Therefore the corresponding tower under this sort of presentation will give:

$$(A_n, A_n^+) \tag{2.3.4}$$

$$:= (\mathcal{O}_K[\pi^{1/p^n}][[u^{1/p^n}, S_1^{1/p^n}, \dots, S_d^{1/p^n}]] [1/u^{1/p^n}] \langle V_1^{\pm 1/p^n}, \dots, V_e^{\pm 1/p^n} \rangle, \tag{2.3.5}$$

$$\mathcal{O}_K[\pi^{1/p^n}][[u^{1/p^n}, S_1^{1/p^n}, \dots, S_d^{1/p^n}]] \langle V_1^{\pm 1/p^n}, \dots, V_e^{\pm 1/p^n} \rangle). \tag{2.3.6}$$

This is certainly a finite étale tower. □

Proposition 2.3.3. *The tower above is perfectoid.*

²⁴Pseudorigid spaces are actually along our generalization in our mind, which are the first kinds of spaces we would like to study in our project before considering more general topological or functional analytic spaces, namely those general spaces whose structure sheaves of simplicial rings carry topologies or norms such as in [BBBK] and [CS2].

Proof. Note that the corresponding pseudorigid space is actually formally of finite type over \mathcal{O}_K namely we have that the ring at the zeroth level is just

$$\mathcal{O}_K[[u, S_1, \dots, S_d]][1/u] \langle V_1^\pm, \dots, V_e^\pm \rangle.$$

Therefore the corresponding tower under this sort of presentation will give:

$$(A_n, A_n^+) \tag{2.3.7}$$

$$:= (\mathcal{O}_K[\pi^{1/p^n}][[u^{1/p^n}, S_1^{1/p^n}, \dots, S_d^{1/p^n}]] [1/u^{1/p^n}] \langle V_1^{\pm 1/p^n}, \dots, V_e^{\pm 1/p^n} \rangle), \tag{2.3.8}$$

$$\mathcal{O}_K[\pi^{1/p^n}][[u^{1/p^n}, S_1^{1/p^n}, \dots, S_d^{1/p^n}]] \langle V_1^{\pm 1/p^n}, \dots, V_e^{\pm 1/p^n} \rangle. \tag{2.3.9}$$

The corresponding ∞ -level is just:

$$A_\infty \tag{2.3.10}$$

$$:= \mathcal{O}_K[\pi^{1/p^\infty}][[u^{1/p^\infty}, S_1^{1/p^\infty}, \dots, S_d^{1/p^\infty}]] [1/u^{1/p^\infty}] \langle V_1^{\pm 1/p^\infty}, \dots, V_e^{\pm 1/p^\infty} \rangle^\wedge, \tag{2.3.11}$$

$$A_\infty^+ \tag{2.3.12}$$

$$:= \mathcal{O}_K[\pi^{1/p^\infty}][[u^{1/p^\infty}, S_1^{1/p^\infty}, \dots, S_d^{1/p^\infty}]] \langle V_1^{\pm 1/p^\infty}, \dots, V_e^{\pm 1/p^\infty} \rangle^\wedge. \tag{2.3.13}$$

These are certainly the corresponding perfectoid rings in the sense of [KL2, Definition 3.3.1], also see [KL2, Lemma 3.3.28]²⁵. \square

Remark 2.3.4. Here the corresponding rings are regarded as the corresponding topological rings instead of Banach rings, but we can certainly consider the corresponding Banach ring structure induced from the linear topology namely really the ∞ -level of this tower is Fontaine perfectoid adic Banach ring which is Tate in the sense of [KL2, Definition 3.1.1].

Proposition 2.3.5. *The tower is weakly decompleting.*

Proof. We give the proof where $K = \mathbb{Q}_p$, which is certainly carried over to more general situation. The corresponding tower is actually weakly decompleting once one considers the corresponding presentation as above for the corresponding ring of definition:

$$A_\infty \tag{2.3.14}$$

$$:= \mathcal{O}_K[\pi^{1/p^\infty}][[u^{1/p^\infty}, S_1^{1/p^\infty}, \dots, S_d^{1/p^\infty}]] [1/u^{1/p^\infty}] \langle V_1^{\pm 1/p^\infty}, \dots, V_e^{\pm 1/p^\infty} \rangle^\wedge, \tag{2.3.15}$$

$$A_\infty^+ \tag{2.3.16}$$

$$:= \mathcal{O}_K[\pi^{1/p^\infty}][[u^{1/p^\infty}, S_1^{1/p^\infty}, \dots, S_d^{1/p^\infty}]] \langle V_1^{\pm 1/p^\infty}, \dots, V_e^{\pm 1/p^\infty} \rangle^\wedge. \tag{2.3.17}$$

This transfers to the positive characteristic situation under tilting:

$$R_\infty \tag{2.3.18}$$

$$:= k_K[\bar{\pi}^{1/p^\infty}][[\bar{u}^{1/p^\infty}, \bar{S}_1^{1/p^\infty}, \dots, \bar{S}_d^{1/p^\infty}]] [1/\bar{u}^{1/p^\infty}] \langle \bar{V}_1^{\pm 1/p^\infty}, \dots, \bar{V}_e^{\pm 1/p^\infty} \rangle^\wedge, \tag{2.3.19}$$

$$R_\infty^+ \tag{2.3.20}$$

$$:= k_K[\bar{\pi}^{1/p^\infty}][[\bar{u}^{1/p^\infty}, \bar{S}_1^{1/p^\infty}, \dots, \bar{S}_d^{1/p^\infty}]] \langle \bar{V}_1^{\pm 1/p^\infty}, \dots, \bar{V}_e^{\pm 1/p^\infty} \rangle^\wedge. \tag{2.3.21}$$

²⁵As in [KL2, Lemma 3.3.28] one takes the original pseudouniformizer, and goes to a suitably high level to achieve some root u' of the function $X^{p^n} - Xu - u$, and eventually the corresponding element u'^p will divide p .

Then one can finish as in [KL2, Lemma 7.1.2] by comparing this with $R_{(H_{\text{degree number}}, H_{\text{degree number}}^+)}$. \square

2.3.2 Pseudorigid Logarithmic Relative Toric Tower

Setting 2.3.6. We consider the following towers as in [KL2, Chapter 5]. First we let A_0 be the ring $\mathcal{O}_K[[u]] \left\langle \frac{\pi^a}{u^b} \right\rangle [1/u] \left\langle T_1^\pm, \dots, T_d^\pm, T_{d+1}^-, \dots, T_f^- \right\rangle$ and we put A_0^+ to be the ring

$$\mathcal{O}_K[[u]] \left\langle \frac{\pi^a}{u^b} \right\rangle \cdot \left\langle T_1^\pm, \dots, T_d^\pm, T_{d+1}^-, \dots, T_f^- \right\rangle. \quad (2.3.22)$$

And for the higher level we have the following rings:

$$(A_n, A_n^+) \quad (2.3.23)$$

$$:= (\mathcal{O}_K(\pi^{1/p^n})[[u]] \left\langle \frac{\pi^a}{u^b} \right\rangle [1/u] (u^{\pm 1/p^n}) \left\langle T_1^{\pm 1/p^n}, \dots, T_d^{\pm 1/p^n}, T_{d+1}^{-1/p^n}, \dots, T_f^{-1/p^n} \right\rangle), \quad (2.3.24)$$

$$\mathcal{O}_K(\pi^{1/p^n})[[u]] \left\langle \frac{\pi^a}{u^b} \right\rangle [u^{1/p^n}] \left\langle T_1^{\pm 1/p^n}, \dots, T_d^{\pm 1/p^n}, T_{d+1}^{-1/p^n}, \dots, T_f^{-1/p^n} \right\rangle), \quad (2.3.25)$$

which implies that we have:

$$A_0 \longrightarrow A_1 \longrightarrow \dots \longrightarrow A_n \longrightarrow \dots, \forall n \geq 0.$$

Proposition 2.3.7. *The tower above is finite étale.*

Proof. Note that the corresponding pseudorigid space is actually formally of finite type over \mathcal{O}_K namely we have that the ring at the zeroth level is just

$$\mathcal{O}_K[[u, S_1, \dots, S_d]][1/u] \left\langle V_1^\pm, \dots, V_e^\pm \right\rangle.$$

Therefore the corresponding tower under this sort of presentation will give:

$$(A_n, A_n^+) \quad (2.3.26)$$

$$:= (\mathcal{O}_K[\pi^{1/p^n}][[u^{1/p^n}, S_1^{1/p^n}, \dots, S_d^{1/p^n}]] [1/u^{1/p^n}] \left\langle V_1^{\pm 1/p^n}, \dots, V_e^{\pm 1/p^n}, T_{e+1}^{-1/p^n}, \dots, T_f^{-1/p^n} \right\rangle), \quad (2.3.27)$$

$$\mathcal{O}_K[\pi^{1/p^n}][[u^{1/p^n}, S_1^{1/p^n}, \dots, S_d^{1/p^n}]] \left\langle V_1^{\pm 1/p^n}, \dots, V_e^{\pm 1/p^n}, T_{e+1}^{-1/p^n}, \dots, T_f^{-1/p^n} \right\rangle). \quad (2.3.28)$$

This is certainly a finite étale tower. \square

Proposition 2.3.8. *The tower above is perfectoid.*

Proof. Note that the corresponding pseudorigid space is actually formally of finite type over \mathcal{O}_K namely we have that the ring at the zeroth level is just

$$\mathcal{O}_K[[u, S_1, \dots, S_d]][1/u] \left\langle V_1^\pm, \dots, V_e^\pm \right\rangle.$$

Therefore the corresponding tower under this sort of presentation will give:

$$(A_n, A_n^+) \quad (2.3.29)$$

$$:= (\mathcal{O}_K[\pi^{1/p^n}][[u^{1/p^n}, S_1^{1/p^n}, \dots, S_d^{1/p^n}]] [1/u^{1/p^n}] \left\langle V_1^{\pm 1/p^n}, \dots, V_e^{\pm 1/p^n}, T_{e+1}^{-1/p^n}, \dots, T_f^{-1/p^n} \right\rangle), \quad (2.3.30)$$

$$\mathcal{O}_K[\pi^{1/p^n}][[u^{1/p^n}, S_1^{1/p^n}, \dots, S_d^{1/p^n}]] \left\langle V_1^{\pm 1/p^n}, \dots, V_e^{\pm 1/p^n}, T_{e+1}^{-1/p^n}, \dots, T_f^{-1/p^n} \right\rangle). \quad (2.3.31)$$

The corresponding ∞ -level is just:

$$A_\infty := \tag{2.3.32}$$

$$\mathcal{O}_K[\pi^{1/p^\infty}][[u^{1/p^\infty}, S_1^{1/p^\infty}, \dots, S_d^{1/p^\infty}][1/u^{1/p^\infty}] \left\langle V_1^{\pm 1/p^\infty}, \dots, V_e^{\pm 1/p^\infty}, T_{e+1}^{-1/p^\infty}, \dots, T_f^{-1/p^\infty} \right\rangle^\wedge, \tag{2.3.33}$$

$$A_\infty^+ \tag{2.3.34}$$

$$:= \mathcal{O}_K[\pi^{1/p^\infty}][[u^{1/p^\infty}, S_1^{1/p^\infty}, \dots, S_d^{1/p^\infty}]] \left\langle V_1^{\pm 1/p^\infty}, \dots, V_e^{\pm 1/p^\infty}, T_{e+1}^{-1/p^\infty}, \dots, T_f^{-1/p^\infty} \right\rangle^\wedge. \tag{2.3.35}$$

These are certainly the corresponding perfectoid rings in the sense of [KL2, Definition 3.3.1], also see [KL2, Lemma 3.3.28]²⁶. \square

Remark 2.3.9. Here the corresponding rings are regarded as the corresponding topological rings instead of Banach rings, but we can certainly consider the corresponding Banach ring structure induced from the linear topology namely really the ∞ -level of this tower is Fontaine perfectoid adic Banach ring which is Tate in the sense of [KL2, Definition 3.1.1].

Proposition 2.3.10. *The tower is weakly decompleting.*

Proof. We give the proof where $K = \mathbb{Q}_p$, which is certainly carried over to more general situation. The corresponding tower is actually weakly decompleting once one considers the corresponding presentation as above for the corresponding ring of definition:

$$A_\infty := \tag{2.3.36}$$

$$\mathcal{O}_K[\pi^{1/p^\infty}][[u^{1/p^\infty}, S_1^{1/p^\infty}, \dots, S_d^{1/p^\infty}][1/u^{1/p^\infty}] \left\langle V_1^{\pm 1/p^\infty}, \dots, V_e^{\pm 1/p^\infty}, T_{e+1}^{-1/p^\infty}, \dots, T_f^{-1/p^\infty} \right\rangle^\wedge, \tag{2.3.37}$$

$$A_\infty^+ \tag{2.3.38}$$

$$:= \mathcal{O}_K[\pi^{1/p^\infty}][[u^{1/p^\infty}, S_1^{1/p^\infty}, \dots, S_d^{1/p^\infty}]] \left\langle V_1^{\pm 1/p^\infty}, \dots, V_e^{\pm 1/p^\infty}, T_{e+1}^{-1/p^\infty}, \dots, T_f^{-1/p^\infty} \right\rangle^\wedge. \tag{2.3.39}$$

This transfers to the positive characteristic situation under tilting:

$$R_\infty := \tag{2.3.40}$$

$$k_K[\overline{\pi}^{1/p^\infty}][[\overline{u}^{1/p^\infty}, \overline{S}_1^{1/p^\infty}, \dots, \overline{S}_d^{1/p^\infty}][1/\overline{u}^{1/p^\infty}] \left\langle \overline{V}_1^{\pm 1/p^\infty}, \dots, \overline{V}_e^{\pm 1/p^\infty}, T_{e+1}^{-1/p^\infty}, \dots, T_f^{-1/p^\infty} \right\rangle^\wedge, \tag{2.3.41}$$

$$R_\infty^+ \tag{2.3.42}$$

$$:= k_K[\overline{\pi}^{1/p^\infty}][[\overline{u}^{1/p^\infty}, \overline{S}_1^{1/p^\infty}, \dots, \overline{S}_d^{1/p^\infty}]] \left\langle \overline{V}_1^{\pm 1/p^\infty}, \dots, \overline{V}_e^{\pm 1/p^\infty}, T_{e+1}^{-1/p^\infty}, \dots, T_f^{-1/p^\infty} \right\rangle^\wedge. \tag{2.3.43}$$

Then one can finish as in [KL2, Lemma 7.1.2] by comparing this with $R_{(H_{\text{degree number}}, H_{\text{degree number}}^+)}$. \square

²⁶As in [KL2, Lemma 3.3.28] one takes the original pseudouniformizer, and goes to a suitably high level to achieve some root u' of the function $X^{p^n} - Xu - u$, and eventually the corresponding element u'^p will divide p .

2.3.3 Robba Sheaves over Pseudorigid Spaces

Now we consider Kedlaya-Liu's Robba sheaves, which are also some motivic and functorial construction. Our space is actually regarded over \mathbb{Z}_p , but note that the corresponding pseudorigid affinoid could be defined over arbitrary integral ring \mathcal{O}_K of some analytic field K . Let X be a pseudorigid space over \mathcal{O}_K , but we regard this as a corresponding Tate adic space over \mathbb{Z}_p . Now we consider the corresponding pro-étale site of X which we denote it as $X_{\text{proét}}$.

Definition 2.3.11. We now apply the corresponding definitions of the Robba rings in [KL2, Definition 4.1.1] with E therein being just \mathbb{Q}_p to any perfectoid subdomain (P_∞, P_∞^+) of $X_{\text{proét}}$. For each such perfectoid, we promote it to be a Banach ring $(P_\infty, P_\infty^+, \|\cdot\|_\infty)$, then we have the corresponding construction of the Robba rings in [KL2, Definition 4.1.1] (we use the notation $\tilde{\Pi}$ to represent the notation \mathcal{R}):

$$\tilde{\Pi}_{(P_\infty, P_\infty^+, \|\cdot\|_\infty)}^{[s,r]} \quad (2.3.44)$$

$$\varprojlim_s \tilde{\Pi}_{(P_\infty, P_\infty^+, \|\cdot\|_\infty)}^{[s,r]} \quad (2.3.45)$$

$$\varinjlim_r \varprojlim_s \tilde{\Pi}_{(P_\infty, P_\infty^+, \|\cdot\|_\infty)}^{[s,r]}. \quad (2.3.46)$$

Then one just organizes these to be certain presheaves over the site $X_{\text{proét}}$:

$$\tilde{\Pi}_{X, \text{proét}}^{[s,r]} \quad (2.3.47)$$

$$\varprojlim_s \tilde{\Pi}_{X, \text{proét}}^{[s,r]} \quad (2.3.48)$$

$$\varinjlim_r \varprojlim_s \tilde{\Pi}_{X, \text{proét}}^{[s,r]}. \quad (2.3.49)$$

However the corresponding construction is not canonical since the promotion to Banach rings locally is not functorial. But we do have the situation that they are actually sheaves due to the fact that we can regard them as sheaves over some preperfectoid spaces under the corresponding tilting of the total spaces, where we use the same notation to denote the sheaves.

$$\tilde{\Pi}_{X, \text{proét}}^{[s,r]} \quad (2.3.50)$$

$$\varprojlim_s \tilde{\Pi}_{X, \text{proét}}^{[s,r]} \quad (2.3.51)$$

$$\varinjlim_r \varprojlim_s \tilde{\Pi}_{X, \text{proét}}^{[s,r]}. \quad (2.3.52)$$

The corresponding total spaces are defined as:

$$\text{Spectrum}_{\text{adic}}^{\text{total}}(\tilde{\Pi}_{X, \text{proét}}^{[s,r]}, \tilde{\Pi}_{X, \text{proét}}^{[s,r],+}), \quad (2.3.53)$$

$$\text{Spectrum}_{\text{adic}}^{\text{total}}(\varprojlim_s \tilde{\Pi}_{X, \text{proét}}^{[s,r]}, \varprojlim_s \tilde{\Pi}_{X, \text{proét}}^{[s,r],+}), \quad (2.3.54)$$

$$\text{Spectrum}_{\text{adic}}^{\text{total}}(\varinjlim_r \varprojlim_s \tilde{\Pi}_{X, \text{proét}}^{[s,r]}, \varinjlim_r \varprojlim_s \tilde{\Pi}_{X, \text{proét}}^{[s,r],+}). \quad (2.3.55)$$

Definition 2.3.12. As in [KL2, Chapter 4.3 and Chapter 8], consider the corresponding total space:

$$\text{Spectrumadic}_{\text{total}}(\tilde{\Pi}_{X,\text{proét}}^{[s,r]}, \tilde{\Pi}_{X,\text{proét}}^{[s,r],+}). \quad (2.3.56)$$

We define the corresponding φ -sheaves (where φ is Frobenius lifting from p -th power Frobenius coming from the characteristic p rings encoded in the construction recalled above from [KL2]) to be sheaves locally attached to étale-stably pseudocoherent sheaves carrying the corresponding semilinear Frobenius morphisms realizing the isomorphisms under pullback. Then over

$$\text{Spectrumadic}_{\text{total}}(\varprojlim_s \tilde{\Pi}_{X,\text{proét}}^{[s,r]}, \varprojlim_s \tilde{\Pi}_{X,\text{proét}}^{[s,r],+}) \quad (2.3.57)$$

we define the similar pseudocoherent sheaves (complete with respect to the natural topology) locally base change to some sheaves in the previous kind. Finally we define the similar pseudocoherent sheaves (complete with respect to the natural topology) over

$$\text{Spectrumadic}_{\text{total}}(\varinjlim_r \varprojlim_s \tilde{\Pi}_{X,\text{proét}}^{[s,r]}, \varinjlim_r \varprojlim_s \tilde{\Pi}_{X,\text{proét}}^{[s,r],+}), \quad (2.3.58)$$

locally base change to some sheaves in the previous kind²⁷.

²⁷Here it is certainly not expected to be the case where we could have uniform radius $r > 0$.

2.3.4 Robba Sheaves over (p, I) -Adic Spaces

Now we consider Kedlaya-Liu's Robba sheaves over more general adic spaces, which is also some motivic and functorial construction. Let X be a Tate adic space over \mathbb{Z}_p where p is assumed to be topologically nilpotent. Now we consider the corresponding pro-étale site of X which we denote it as $X_{\text{proét}}$.

Definition 2.3.13. We now apply the corresponding definitions of the Robba rings in [KL2, Definition 4.1.1] with E therein being just \mathbb{Q}_p to any perfectoid subdomain (P_∞, P_∞^+) of $X_{\text{proét}}$. For each such perfectoid, we promote it to be a Banach ring $(P_\infty, P_\infty^+, \|\cdot\|_\infty)$, then we have the corresponding construction of the Robba rings in [KL2, Definition 4.1.1] (we use the notation $\tilde{\Pi}$ to represent the notation \mathcal{R}):

$$\tilde{\Pi}_{(P_\infty, P_\infty^+, \|\cdot\|_\infty)}^{[s,r]} \quad (2.3.59)$$

$$\varprojlim_s \tilde{\Pi}_{(P_\infty, P_\infty^+, \|\cdot\|_\infty)}^{[s,r]} \quad (2.3.60)$$

$$\varinjlim_r \varprojlim_s \tilde{\Pi}_{(P_\infty, P_\infty^+, \|\cdot\|_\infty)}^{[s,r]}. \quad (2.3.61)$$

Then one just organizes these to be certain presheaves over the site $X_{\text{proét}}$:

$$\tilde{\Pi}_{X, \text{proét}}^{[s,r]} \quad (2.3.62)$$

$$\varprojlim_s \tilde{\Pi}_{X, \text{proét}}^{[s,r]} \quad (2.3.63)$$

$$\varinjlim_r \varprojlim_s \tilde{\Pi}_{X, \text{proét}}^{[s,r]}. \quad (2.3.64)$$

However the corresponding construction is not canonical since the promotion to Banach rings locally is not functorial. But we do have the situation that they are actually sheaves due to the fact that we can regard them as sheaves over some preperfectoid spaces under the corresponding tilting of the total spaces, where we use the same notation to denote the sheaves.

$$\tilde{\Pi}_{X, \text{proét}}^{[s,r]} \quad (2.3.65)$$

$$\varprojlim_s \tilde{\Pi}_{X, \text{proét}}^{[s,r]} \quad (2.3.66)$$

$$\varinjlim_r \varprojlim_s \tilde{\Pi}_{X, \text{proét}}^{[s,r]}. \quad (2.3.67)$$

The corresponding total spaces are defined as:

$$\text{Spectrumadic}_{\text{total}}(\tilde{\Pi}_{X, \text{proét}}^{[s,r]}, \tilde{\Pi}_{X, \text{proét}}^{[s,r],+}), \quad (2.3.68)$$

$$\text{Spectrumadic}_{\text{total}}(\varprojlim_s \tilde{\Pi}_{X, \text{proét}}^{[s,r]}, \varprojlim_s \tilde{\Pi}_{X, \text{proét}}^{[s,r],+}), \quad (2.3.69)$$

$$\text{Spectrumadic}_{\text{total}}(\varinjlim_r \varprojlim_s \tilde{\Pi}_{X, \text{proét}}^{[s,r]}, \varinjlim_r \varprojlim_s \tilde{\Pi}_{X, \text{proét}}^{[s,r],+}). \quad (2.3.70)$$

Definition 2.3.14. As in [KL2, Chapter 4.3 and Chapter 8], consider the corresponding total space:

$$\text{Spectrumadic}_{\text{total}}(\tilde{\Pi}_{X, \text{proét}}^{[s,r]}, \tilde{\Pi}_{X, \text{proét}}^{[s,r],+}). \quad (2.3.71)$$

We define the corresponding φ -sheaves (where φ is Frobenius lifting from p -th power Frobenius coming from the characteristic p rings encoded in the construction recalled above from [KL2]) to be sheaves locally attached to étale-stably pseudocoherent sheaves carrying the corresponding semilinear Frobenius morphism realizing the isomorphism under pullback. Then over

$$\text{Spectrumadic}_{\text{total}} \left(\varprojlim_s \tilde{\Pi}_{X, \text{proét}}^{[s,r]}, \varprojlim_s \tilde{\Pi}_{X, \text{proét}}^{[s,r],+} \right) \quad (2.3.72)$$

we define the similar pseudocoherent sheaves (complete with respect to the natural topology) locally base change to some sheaves in the previous kind. Finally we define the similar pseudocoherent sheaves (complete with respect to the natural topology) over

$$\text{Spectrumadic}_{\text{total}} \left(\varinjlim_r \varprojlim_s \tilde{\Pi}_{X, \text{proét}}^{[s,r]}, \varinjlim_r \varprojlim_s \tilde{\Pi}_{X, \text{proét}}^{[s,r],+} \right), \quad (2.3.73)$$

locally base change to some sheaves in the previous kind²⁸.

Remark 2.3.15. The corresponding sheaves of the full Robba rings are well-defined since one could regard them as structure sheaves of some total spaces.

²⁸Here it is certainly not expected to be the case where we could have uniform radius $r > 0$.

2.4 Derived I -Complete THH and HH of Derived I -Complete Rings

2.4.1 Derived I -Complete Objects

The corresponding THH and HH of derived I -complete rings are topological constructions which are very closely related to the corresponding relative p -adic motives. And certain derived I -complete versions are also relevant in some highly nontrivial way. We make some discussion closely after [BMS, Section 2.2, Section 2.3], [BS1] and [NS, Chapter 3]. We now consider the following rings:

Setting 2.4.1. We consider the derived I -complete \mathbb{E}_1 -rings. For instance one can consider some derived I -complete \mathbb{E}_1 -rings relative to the corresponding integral pseudorigid disc. In the following I will be some two sided ideal in the π_0 of the base spectrum R . We regard all the ring spectra as being in the ∞ -category of all ring spectra over R and in the derived ∞ -category $\mathbb{D}(R)$ of all the module spectra over R . We regard all the derived I -complete ring spectra as being in the ∞ -category of all derived I -complete ring spectra over R and in the derived ∞ -category $\mathbb{D}_I(R)$ of all the derived I -complete module spectra over R . We assume that I is central in the noncommutative setting.

Definition 2.4.2. For any such ring R which is assumed to be \mathbb{E}_∞ , we will use the corresponding notations $LTHH(R)$ and $LHH(R)$ to denote the corresponding left Kan extended THH or the corresponding HH functor from [NS, Chapter 3]. And we will use the corresponding notations $LTHH(R)_I$ and $LHH(R)_I$ to denote the corresponding left Kan extended THH or the corresponding HH functor after taking the corresponding derived I -completion as in [NS, Chapter 3].

Definition 2.4.3. For any such ring R which is assumed to be \mathbb{E}_1 , we will use the corresponding notations $THH(R)$ and $HH(R)$ to denote the corresponding THH or the corresponding HH functor from [NS, Chapter 3]. And we will use the corresponding notations $THH(R)_I$ and $HH(R)_I$ to denote the corresponding THH or the corresponding HH functor after taking the corresponding derived I -completion as in [NS, Chapter 3] and [BS1, Chapter 1 Notation].

Example 2.4.4. For instance one considers the adic rings in [FK, Section 1.4], one takes such a ring R which is complete with respect to a corresponding two sided finitely generated ideal I , then the corresponding construction above namely $LTHH(R)_I$ and $LHH(R)_I$ will also be basically topological versions of the spectra in [ELS, Definition 1.3, Definition 1.4] in the corresponding derived noncommutative deformation theory.

As in [ELS, Chapter 5] and more generally, one has the corresponding noncommutative version of the corresponding relative Kähler differential complexes $\text{deRham}_{\text{noncommutative}, B/A}^{\text{degreenumber}}$. For instance if we have the corresponding map $A \rightarrow B$ of adic rings in [FK, Section 1.4] we could then take the corresponding derived I -completion from $\text{deRham}_{\text{noncommutative}, B/A}^{\text{degreenumber}}$ to achieve the corresponding topological version $\text{deRham}_{\text{noncommutative}, B/A, \text{topo}}^{\text{degreenumber}}$.

Remark 2.4.5. As mentioned in the introduction, the corresponding topological derived I -adic version of the corresponding $LTHH$ and LHH spectra should be interesting to study, as over a quasisyntomic site of a quasisyntomic ring Q , [BMS, Proposition 5.15] shows that the corresponding sheaf $\pi_0 \text{HC}^-(-/Q)_p$ will be closely related to the corresponding p -adic complete

and Hodge-complete derived de Rham sheaf over the same site after taking the corresponding unfolding through the quasiregular semiperfectoids. We do not know if this relation could hold in some sense if we consider derived I -adic rings, but one might want to believe that in the situations we considered before the story should be in some sense easier to be established.

Chapter 3

Functional Analytic Theory

3.1 Functional Analytic André-Quillen Homology and Topological Derived de Rham Complexes

3.1.1 The Construction from Kelly-Kremnizer-Mukherjee

We now work in the corresponding foundations from [BBBK], [BBK], [BBM], [BK] and [KKM], namely what we are going to consider will be literally the following $(\infty, 1)$ -categories and the associated constructions with some fixed Banach ring R or \mathbb{F}_1 ¹:

$$\text{Object}_{E_\infty\text{commutativealgebra},\text{Simplicial}}(\text{IndSNorm}_R), \quad (3.1.1)$$

$$\text{Object}_{E_\infty\text{commutativealgebra},\text{Simplicial}}(\text{Ind}^m\text{SNorm}_R), \quad (3.1.2)$$

$$\text{Object}_{E_\infty\text{commutativealgebra},\text{Simplicial}}(\text{IndNorm}_R), \quad (3.1.3)$$

$$\text{Object}_{E_\infty\text{commutativealgebra},\text{Simplicial}}(\text{Ind}^m\text{Norm}_R), \quad (3.1.4)$$

$$\text{Object}_{E_\infty\text{commutativealgebra},\text{Simplicial}}(\text{IndBan}_R), \quad (3.1.5)$$

$$\text{Object}_{E_\infty\text{commutativealgebra},\text{Simplicial}}(\text{Ind}^m\text{Ban}_R), \quad (3.1.6)$$

with

$$\text{Object}_{E_\infty\text{commutativealgebra},\text{Simplicial}}(\text{IndSNorm}_{\mathbb{F}_1}), \quad (3.1.7)$$

$$\text{Object}_{E_\infty\text{commutativealgebra},\text{Simplicial}}(\text{Ind}^m\text{SNorm}_{\mathbb{F}_1}), \quad (3.1.8)$$

$$\text{Object}_{E_\infty\text{commutativealgebra},\text{Simplicial}}(\text{IndNorm}_{\mathbb{F}_1}), \quad (3.1.9)$$

$$\text{Object}_{E_\infty\text{commutativealgebra},\text{Simplicial}}(\text{Ind}^m\text{Norm}_{\mathbb{F}_1}), \quad (3.1.10)$$

$$\text{Object}_{E_\infty\text{commutativealgebra},\text{Simplicial}}(\text{IndBan}_{\mathbb{F}_1}), \quad (3.1.11)$$

$$\text{Object}_{E_\infty\text{commutativealgebra},\text{Simplicial}}(\text{Ind}^m\text{Ban}_{\mathbb{F}_1}). \quad (3.1.12)$$

We then use the notation \mathcal{X} to denote any of these. We now first discuss the corresponding topological version of André-Quillen Homology and the corresponding topological version of Derived de Rham complex parallel to [An1, Chapitre 3], [An2], [B1, Chapter 2, Chapter 8], [Bei,

¹At this moment we do not work over more general rings such as \mathbb{Z} , but this is crucial in the corresponding globalization. That being said, working over \mathbb{F}_1 is in some sense more general even than these situations.

Chapter 1], [G1, Chapter 5], [GL, Chapter 3, Chapter 4], [III1, Chapitre II, Chapitre III], [III2, Chapitre VIII], [Qui, Section 4]. We would like to start from the corresponding context of [KKM, Section 5.2.1, Definition 5.5, Proposition 5.6, Definition 5.8, Definition 5.9]², and represent the construction for the convenience of the readers³. Namely as in [KKM, Section 5.2.1, Definition 5.5, Proposition 5.6] we consider the corresponding cotangent complex associated to any pair object

$$(A, B) \in \mathcal{X} \times \mathcal{X}_A$$

is defined to be (as in [KKM, Section 5.2.1, Definition 5.5, Proposition 5.6]) just:

$$\mathbb{L}_{B/A, \text{KKM}} := \text{deRham}_{A[B]^{\text{degreenumber}/A, \text{KKM}}}^1 \otimes_{A[B]^{\text{degreenumber}}} B. \quad (3.1.13)$$

The corresponding topological André-Quillen homologies (as in [KKM, Section 5.2.1, Definition 5.5, Proposition 5.6]) are defined to be:

$$H_{\text{degreenumber}, \text{AQ}, \text{KKM}} := \pi_{\text{degreenumber}}(\mathbb{L}_{B/A, \text{KKM}}). \quad (3.1.14)$$

We then as in the definition [KKM, Definition 5.8, Definition 5.9, Section 5.2.1] have the corresponding different kinds of $(\infty, 1)$ -rings of de Rham complexes and we denote that by:

$$\text{Kan}_{\text{Left}} \text{deRham}_{B/A, \text{functionalanalytic}, \text{KKM}}^{\text{degreenumber}}, \text{Kan}_{\text{Left}} \text{Fil}_{\text{deRham}_{B/A, \text{functionalanalytic}, \text{KKM}}^{\text{degreenumber}}}^*. \quad (3.1.15)$$

Then we need to take the corresponding Hodge-Filtered completion by using the corresponding filtration associated as above:

$$\text{Kan}_{\text{Left}} \widehat{\text{deRham}}_{B/A, \text{functionalanalytic}, \text{KKM}}^{\text{degreenumber}}, \text{Kan}_{\text{Left}} \widehat{\text{Fil}}_{\text{deRham}_{B/A, \text{functionalanalytic}, \text{KKM}}^{\text{degreenumber}}}^*. \quad (3.1.16)$$

²Also see [Ra, Definition 4.4.7, Construction 4.4.10, Theorem 5.3.6, Definition 5.2.4, Corollary 5.3.9].

³Here we just present the homotopical contexts in Kelly-Kremnizer-Mukherjee for the general $(\infty, 1)$ -categories as above.

3.2 Functional Analytic Derived Prismatic Cohomology and Functional Analytic Derived Perfectoidizations

3.2.1 General Constructions

Lurie's book [Lu3, Section 5.5.8] illustrates in a very general way the corresponding constructions on how we could extend in a certain (∞, n) -category the corresponding nonabelian derived functor from a smaller class of generators in some discrete sense to a large (∞, n) -categorical closure by considering enough colimits being sifted. Therefore in our situation by using the foundation in [BBBK], [BBK], [BBM], [BK] and [KKM] one can define and extend from Tate series rings, Stein series rings, formal series rings and dagger series rings (as those in [BBM, Section 4.2]) into very large $(\infty, 1)$ -categories in the following sense:

$$\mathrm{Object}_{E_\infty\text{commutativealgebra}, \mathrm{Simplicial}}(\mathrm{IndSNorm}_R), \quad (3.2.1)$$

$$\mathrm{Object}_{E_\infty\text{commutativealgebra}, \mathrm{Simplicial}}(\mathrm{Ind}^m\mathrm{SNorm}_R), \quad (3.2.2)$$

$$\mathrm{Object}_{E_\infty\text{commutativealgebra}, \mathrm{Simplicial}}(\mathrm{IndNorm}_R), \quad (3.2.3)$$

$$\mathrm{Object}_{E_\infty\text{commutativealgebra}, \mathrm{Simplicial}}(\mathrm{Ind}^m\mathrm{Norm}_R), \quad (3.2.4)$$

$$\mathrm{Object}_{E_\infty\text{commutativealgebra}, \mathrm{Simplicial}}(\mathrm{IndBan}_R), \quad (3.2.5)$$

$$\mathrm{Object}_{E_\infty\text{commutativealgebra}, \mathrm{Simplicial}}(\mathrm{Ind}^m\mathrm{Ban}_R), \quad (3.2.6)$$

with

$$\mathrm{Object}_{E_\infty\text{commutativealgebra}, \mathrm{Simplicial}}(\mathrm{IndSNorm}_{\mathbb{F}_1}), \quad (3.2.7)$$

$$\mathrm{Object}_{E_\infty\text{commutativealgebra}, \mathrm{Simplicial}}(\mathrm{Ind}^m\mathrm{SNorm}_{\mathbb{F}_1}), \quad (3.2.8)$$

$$\mathrm{Object}_{E_\infty\text{commutativealgebra}, \mathrm{Simplicial}}(\mathrm{IndNorm}_{\mathbb{F}_1}), \quad (3.2.9)$$

$$\mathrm{Object}_{E_\infty\text{commutativealgebra}, \mathrm{Simplicial}}(\mathrm{Ind}^m\mathrm{Norm}_{\mathbb{F}_1}), \quad (3.2.10)$$

$$\mathrm{Object}_{E_\infty\text{commutativealgebra}, \mathrm{Simplicial}}(\mathrm{IndBan}_{\mathbb{F}_1}), \quad (3.2.11)$$

$$\mathrm{Object}_{E_\infty\text{commutativealgebra}, \mathrm{Simplicial}}(\mathrm{Ind}^m\mathrm{Ban}_{\mathbb{F}_1}). \quad (3.2.12)$$

Definition 3.2.1. Since we are going to consider Bhatt-Scholze's derived prismatic functors [BS1, Construction 7.6], so we take the formal series over a general prism (A, I) (assume this to be bounded) such that A/I is also Banach. Then we consider the corresponding formal series ring over A/I , then take the corresponding projection resolution compactly generated closure of them to get the sub $(\infty, 1)$ -categories of above $(\infty, 1)$ -categories, which we are going to denote them as:

$$\mathrm{Object}_{E_\infty\text{commutativealgebra}, \mathrm{Simplicial}}(\mathrm{IndSNorm}_{A/I})^{\mathrm{smoothformalseriesclosure}}, \quad (3.2.13)$$

$$\mathrm{Object}_{E_\infty\text{commutativealgebra}, \mathrm{Simplicial}}(\mathrm{Ind}^m\mathrm{SNorm}_{A/I})^{\mathrm{smoothformalseriesclosure}}, \quad (3.2.14)$$

$$\mathrm{Object}_{E_\infty\text{commutativealgebra}, \mathrm{Simplicial}}(\mathrm{IndNorm}_{A/I})^{\mathrm{smoothformalseriesclosure}}, \quad (3.2.15)$$

$$\mathrm{Object}_{E_\infty\text{commutativealgebra}, \mathrm{Simplicial}}(\mathrm{Ind}^m\mathrm{Norm}_{A/I})^{\mathrm{smoothformalseriesclosure}}, \quad (3.2.16)$$

$$\mathrm{Object}_{E_\infty\text{commutativealgebra}, \mathrm{Simplicial}}(\mathrm{IndBan}_{A/I})^{\mathrm{smoothformalseriesclosure}}, \quad (3.2.17)$$

$$\mathrm{Object}_{E_\infty\text{commutativealgebra}, \mathrm{Simplicial}}(\mathrm{Ind}^m\mathrm{Ban}_{A/I})^{\mathrm{smoothformalseriesclosure}}, \quad (3.2.18)$$

as in [BBM, Section 4.2] on the corresponding analytification defined from extension from formal series rings.

Consequently one applies this idea directly to Bhatt-Scholze's prismatic construction [BS1], one directly gets the corresponding functional analytic $(\infty, 1)$ -categorical prismatic cohomologies.

3.2.2 ∞ -Categorical Functional Analytic Prismatic Cohomologies and ∞ -Categorical Functional Analytic Preperfectoidizations

Definition 3.2.2. One can actually define the derived prismatic cohomologies through derived topological Hochschild cohomologies, derived topological period cohomologies and derived topological cyclic cohomologies as in [BMS, Section 2.2, Section 2.3], [BS1, Theorem 1.13]:

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{THH}, \mathrm{Kan}_{\mathrm{Left}}\mathrm{TP}, \mathrm{Kan}_{\mathrm{Left}}\mathrm{TC}, \quad (3.2.19)$$

on the following $(\infty, 1)$ -compactly generated closures of the corresponding polynomials⁴ given over A/I with a chosen prism (A, I) ⁵:

⁴Definitely, we need to put certain norms over in some relatively canonical way, as in [BBM, Section 4.2] one can basically consider rigid ones and dagger ones, and so on. We restrict to the *formal* one. This means that we are going to consider the corresponding p -adic topology only such as the norm over $A/I \langle T_1, \dots, T_n \rangle$, while rigid analytic situation is usually over a field such as in [G2] over $B_{\mathrm{dR}} \langle T_1, \dots, T_n \rangle$ for instance. One can then in the same way define the corresponding:

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{THH}, \mathrm{Kan}_{\mathrm{Left}}\mathrm{TP}, \mathrm{Kan}_{\mathrm{Left}}\mathrm{TC}, \quad (3.2.20)$$

over

$$\mathrm{Object}_{\mathrm{E}_{\infty}\text{commutativealgebra}, \mathrm{Simplicial}}(\mathrm{IndSNorm}_{B_{\mathrm{dR}}})^{\text{smoothformalseriesclosure}}, \quad (3.2.21)$$

$$\mathrm{Object}_{\mathrm{E}_{\infty}\text{commutativealgebra}, \mathrm{Simplicial}}(\mathrm{Ind}^m\mathrm{SNorm}_{B_{\mathrm{dR}}})^{\text{smoothformalseriesclosure}}, \quad (3.2.22)$$

$$\mathrm{Object}_{\mathrm{E}_{\infty}\text{commutativealgebra}, \mathrm{Simplicial}}(\mathrm{IndNorm}_{B_{\mathrm{dR}}})^{\text{smoothformalseriesclosure}}, \quad (3.2.23)$$

$$\mathrm{Object}_{\mathrm{E}_{\infty}\text{commutativealgebra}, \mathrm{Simplicial}}(\mathrm{Ind}^m\mathrm{Norm}_{B_{\mathrm{dR}}})^{\text{smoothformalseriesclosure}}, \quad (3.2.24)$$

$$\mathrm{Object}_{\mathrm{E}_{\infty}\text{commutativealgebra}, \mathrm{Simplicial}}(\mathrm{IndBan}_{B_{\mathrm{dR}}})^{\text{smoothformalseriesclosure}}, \quad (3.2.25)$$

$$\mathrm{Object}_{\mathrm{E}_{\infty}\text{commutativealgebra}, \mathrm{Simplicial}}(\mathrm{Ind}^m\mathrm{Ban}_{B_{\mathrm{dR}}})^{\text{smoothformalseriesclosure}}, \quad (3.2.26)$$

by

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{THH}_{\text{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \text{formalanalytification}}(\mathcal{O}) := \quad (3.2.27)$$

$$\left(\underset{i}{\text{homotopycolimit}} \underset{\text{sifted, derivedcategory}_{\infty}(B_{\mathrm{dR}}\text{-Module})}{\mathrm{Kan}_{\mathrm{Left}}\mathrm{THH}_{\text{functionalanalytic}, \mathrm{KKM}}(\mathcal{O}_i)} \right)_{\mathrm{BBM}, \text{formalanalytification}}, \quad (3.2.28)$$

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{TP}_{\text{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \text{formalanalytification}}(\mathcal{O}) := \quad (3.2.29)$$

$$\left(\underset{i}{\text{homotopycolimit}} \underset{\text{sifted, derivedcategory}_{\infty}(B_{\mathrm{dR}}\text{-Module})}{\mathrm{Kan}_{\mathrm{Left}}\mathrm{TP}_{\text{functionalanalytic}, \mathrm{KKM}}(\mathcal{O}_i)} \right)_{\mathrm{BBM}, \text{formalanalytification}}, \quad (3.2.30)$$

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{TC}_{\text{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \text{formalanalytification}}(\mathcal{O}) := \quad (3.2.31)$$

$$\left(\underset{i}{\text{homotopycolimit}} \underset{\text{sifted, derivedcategory}_{\infty}(B_{\mathrm{dR}}\text{-Module})}{\mathrm{Kan}_{\mathrm{Left}}\mathrm{TC}_{\text{functionalanalytic}, \mathrm{KKM}}(\mathcal{O}_i)} \right)_{\mathrm{BBM}, \text{formalanalytification}}, \quad (3.2.32)$$

where each \mathcal{O}_i is given as some $B_{\mathrm{dR}} \langle T_1, \dots, T_n \rangle$. See [G2], [G1].

⁵In all the following, we assume this prism to be bounded and satisfy that A/I is Banach.

$$\text{Object}_{E_\infty\text{-commutativealgebra,Simplicial}}(\text{IndSNorm}_{A/I})^{\text{smoothformalseriesclosure}}, \quad (3.2.33)$$

$$\text{Object}_{E_\infty\text{-commutativealgebra,Simplicial}}(\text{Ind}^m\text{SNorm}_{A/I})^{\text{smoothformalseriesclosure}}, \quad (3.2.34)$$

$$\text{Object}_{E_\infty\text{-commutativealgebra,Simplicial}}(\text{IndNorm}_{A/I})^{\text{smoothformalseriesclosure}}, \quad (3.2.35)$$

$$\text{Object}_{E_\infty\text{-commutativealgebra,Simplicial}}(\text{Ind}^m\text{Norm}_{A/I})^{\text{smoothformalseriesclosure}}, \quad (3.2.36)$$

$$\text{Object}_{E_\infty\text{-commutativealgebra,Simplicial}}(\text{IndBan}_{A/I})^{\text{smoothformalseriesclosure}}, \quad (3.2.37)$$

$$\text{Object}_{E_\infty\text{-commutativealgebra,Simplicial}}(\text{Ind}^m\text{Ban}_{A/I})^{\text{smoothformalseriesclosure}}. \quad (3.2.38)$$

We call the corresponding functors are derived functional analytic Hochschild cohomologies, derived functional analytic period cohomologies and derived functional analytic cyclic cohomologies, which we are going to denote them as in the following:

$$\text{Kan}_{\text{Left}}\text{THH}_{\text{functionalanalytic,KKM,BBM,formalanalytification}}(\mathcal{O}) := \quad (3.2.39)$$

$$\left(\text{homotopycolimit}_{i \text{ sifted, derivedcategory}_\infty(A/I\text{-Module})} \text{Kan}_{\text{Left}}\text{THH}_{\text{functionalanalytic,KKM}}(\mathcal{O}_i) \right) \quad (3.2.40)$$

$$\text{)BBM,formalanalytification} \quad (3.2.41)$$

$$\text{Kan}_{\text{Left}}\text{TP}_{\text{functionalanalytic,KKM,BBM,formalanalytification}}(\mathcal{O}) := \quad (3.2.42)$$

$$\left(\text{homotopycolimit}_{i \text{ sifted, derivedcategory}_\infty(A/I\text{-Module})} \text{Kan}_{\text{Left}}\text{TP}_{\text{functionalanalytic,KKM}}(\mathcal{O}_i) \right) \quad (3.2.43)$$

$$\text{)BBM,formalanalytification} \quad (3.2.44)$$

$$\text{Kan}_{\text{Left}}\text{TC}_{\text{functionalanalytic,KKM,BBM,formalanalytification}}(\mathcal{O}) := \quad (3.2.45)$$

$$\left(\text{homotopycolimit}_{i \text{ sifted, derivedcategory}_\infty(A/I\text{-Module})} \text{Kan}_{\text{Left}}\text{TC}_{\text{functionalanalytic,KKM}}(\mathcal{O}_i) \right) \quad (3.2.46)$$

$$\text{)BBM,formalanalytification} \quad (3.2.47)$$

by writing any object \mathcal{O} as the corresponding colimit

$$\text{homotopycolimit}_{i \text{ sifted}} \mathcal{O}_i.$$

These are quite large $(\infty, 1)$ -commutative ring objects in the corresponding $(\infty, 1)$ -categories for $R = A/I$:

$$\text{Object}_{E_\infty\text{-commutativealgebra,Simplicial}}(\text{IndSNorm}_R), \quad (3.2.48)$$

$$\text{Object}_{E_\infty\text{-commutativealgebra,Simplicial}}(\text{Ind}^m\text{SNorm}_R), \quad (3.2.49)$$

$$\text{Object}_{E_\infty\text{-commutativealgebra,Simplicial}}(\text{IndNorm}_R), \quad (3.2.50)$$

$$\text{Object}_{E_\infty\text{-commutativealgebra,Simplicial}}(\text{Ind}^m\text{Norm}_R), \quad (3.2.51)$$

$$\text{Object}_{E_\infty\text{-commutativealgebra,Simplicial}}(\text{IndBan}_R), \quad (3.2.52)$$

$$\text{Object}_{E_\infty\text{-commutativealgebra,Simplicial}}(\text{Ind}^m\text{Ban}_R), \quad (3.2.53)$$

after taking the formal series ring left Kan extension analytification from [BBM, Section 4.2], which is defined by taking the left Kan extension to all the $(\infty, 1)$ -ring objects in the ∞ -derived category of all A -modules from formal series rings over A , into:

$$\mathrm{Object}_{E_{\infty}\mathrm{commutativealgebra},\mathrm{Simplicial}}(\mathrm{IndSNorm}_{\mathbb{F}_1})_{A'} \quad (3.2.54)$$

$$\mathrm{Object}_{E_{\infty}\mathrm{commutativealgebra},\mathrm{Simplicial}}(\mathrm{Ind}^m\mathrm{SNorm}_{\mathbb{F}_1})_{A'} \quad (3.2.55)$$

$$\mathrm{Object}_{E_{\infty}\mathrm{commutativealgebra},\mathrm{Simplicial}}(\mathrm{IndNorm}_{\mathbb{F}_1})_{A'} \quad (3.2.56)$$

$$\mathrm{Object}_{E_{\infty}\mathrm{commutativealgebra},\mathrm{Simplicial}}(\mathrm{Ind}^m\mathrm{Norm}_{\mathbb{F}_1})_{A'} \quad (3.2.57)$$

$$\mathrm{Object}_{E_{\infty}\mathrm{commutativealgebra},\mathrm{Simplicial}}(\mathrm{IndBan}_{\mathbb{F}_1})_{A'} \quad (3.2.58)$$

$$\mathrm{Object}_{E_{\infty}\mathrm{commutativealgebra},\mathrm{Simplicial}}(\mathrm{Ind}^m\mathrm{Ban}_{\mathbb{F}_1})_{A'} \quad (3.2.59)$$

Remark 3.2.3. One should actually do this in a more coherent way as in [Ra], [NS], [KKM], by applying directly the corresponding construction to objects in the corresponding $(\infty, 1)$ -objects above, even the corresponding A_{∞} -objects. The constructions here are the corresponding functional analytic counterpart of the corresponding condensed constructions in [M] after [CS1] and [CS2].

Definition 3.2.4. Then we can in the same fashion consider the corresponding derived prismatic complexes [BS1, Construction 7.6]⁶ for the commutative algebras as in the above (for a given prism (A, I)):

$$\mathrm{Kan}_{\mathrm{Left}}\Delta_{?/A'} \quad (3.2.60)$$

by the regular corresponding left Kan extension techniques on the following $(\infty, 1)$ -compactly generated closures of the corresponding polynomials given over A/I with a chosen prism (A, I) :

$$\mathrm{Object}_{E_{\infty}\mathrm{commutativealgebra},\mathrm{Simplicial}}(\mathrm{IndSNorm}_{A/I})^{\mathrm{smoothformalseriesclosure}}, \quad (3.2.61)$$

$$\mathrm{Object}_{E_{\infty}\mathrm{commutativealgebra},\mathrm{Simplicial}}(\mathrm{Ind}^m\mathrm{SNorm}_{A/I})^{\mathrm{smoothformalseriesclosure}}, \quad (3.2.62)$$

$$\mathrm{Object}_{E_{\infty}\mathrm{commutativealgebra},\mathrm{Simplicial}}(\mathrm{IndNorm}_{A/I})^{\mathrm{smoothformalseriesclosure}}, \quad (3.2.63)$$

$$\mathrm{Object}_{E_{\infty}\mathrm{commutativealgebra},\mathrm{Simplicial}}(\mathrm{Ind}^m\mathrm{Norm}_{A/I})^{\mathrm{smoothformalseriesclosure}}, \quad (3.2.64)$$

$$\mathrm{Object}_{E_{\infty}\mathrm{commutativealgebra},\mathrm{Simplicial}}(\mathrm{IndBan}_{A/I})^{\mathrm{smoothformalseriesclosure}}, \quad (3.2.65)$$

$$\mathrm{Object}_{E_{\infty}\mathrm{commutativealgebra},\mathrm{Simplicial}}(\mathrm{Ind}^m\mathrm{Ban}_{A/I})^{\mathrm{smoothformalseriesclosure}}. \quad (3.2.66)$$

We call the corresponding functors functional analytic derived prismatic complexes which we are going to denote that as in the following:

$$\mathrm{Kan}_{\mathrm{Left}}\Delta_{?/A,\mathrm{functionalanalytic},\mathrm{KKM},\mathrm{BBM},\mathrm{formalanalytification}} \quad (3.2.67)$$

⁶One just applies [BS1, Construction 7.6] and then takes the left Kan extensions.

This would mean the following definition:

$$\mathbf{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, KKM, BBM, formalanalytification}}(\mathcal{O}) \quad (3.2.68)$$

$$:= \left(\underset{i}{\text{homotopycolimit}} \underset{\text{sifted, derivedcategory}_\infty(A/I\text{-Module})}{\mathbf{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, KKM}}(\mathcal{O}_i)} \right)^\wedge \quad (3.2.69)$$

$$\text{)BBM, formalanalytification} \quad (3.2.70)$$

⁷by writing any object \mathcal{O} as the corresponding colimit

$$\underset{i}{\text{homotopycolimit}} \underset{\text{sifted}}{\mathcal{O}_i}.$$

These are quite large $(\infty, 1)$ -commutative ring objects in the corresponding $(\infty, 1)$ -categories for $R = A/I$:

$$\text{Object}_{\text{E}_\infty\text{commutativealgebra, Simplicial}}(\text{IndSNorm}_R), \quad (3.2.71)$$

$$\text{Object}_{\text{E}_\infty\text{commutativealgebra, Simplicial}}(\text{Ind}^m\text{SNorm}_R), \quad (3.2.72)$$

$$\text{Object}_{\text{E}_\infty\text{commutativealgebra, Simplicial}}(\text{IndNorm}_R), \quad (3.2.73)$$

$$\text{Object}_{\text{E}_\infty\text{commutativealgebra, Simplicial}}(\text{Ind}^m\text{Norm}_R), \quad (3.2.74)$$

$$\text{Object}_{\text{E}_\infty\text{commutativealgebra, Simplicial}}(\text{IndBan}_R), \quad (3.2.75)$$

$$\text{Object}_{\text{E}_\infty\text{commutativealgebra, Simplicial}}(\text{Ind}^m\text{Ban}_R), \quad (3.2.76)$$

$$(3.2.77)$$

after taking the formal series ring left Kan extension analytification from [BBM, Section 4.2], which is defined by taking the left Kan extension to all the $(\infty, 1)$ -ring objects in the ∞ -derived category of all A -modules from formal series rings over A , into:

$$\text{Object}_{\text{E}_\infty\text{commutativealgebra, Simplicial}}(\text{IndSNorm}_{\mathbb{F}_1})_{A'}, \quad (3.2.78)$$

$$\text{Object}_{\text{E}_\infty\text{commutativealgebra, Simplicial}}(\text{Ind}^m\text{SNorm}_{\mathbb{F}_1})_{A'}, \quad (3.2.79)$$

$$\text{Object}_{\text{E}_\infty\text{commutativealgebra, Simplicial}}(\text{IndNorm}_{\mathbb{F}_1})_{A'}, \quad (3.2.80)$$

$$\text{Object}_{\text{E}_\infty\text{commutativealgebra, Simplicial}}(\text{Ind}^m\text{Norm}_{\mathbb{F}_1})_{A'}, \quad (3.2.81)$$

$$\text{Object}_{\text{E}_\infty\text{commutativealgebra, Simplicial}}(\text{IndBan}_{\mathbb{F}_1})_{A'}, \quad (3.2.82)$$

$$\text{Object}_{\text{E}_\infty\text{commutativealgebra, Simplicial}}(\text{Ind}^m\text{Ban}_{\mathbb{F}_1})_{A'}. \quad (3.2.83)$$

Then as in [BS1, Definition 8.2] we consider the corresponding perfectoidization in this analytic setting.

⁷Before the Ben-Bassat-Mukherjee p -adic formal analytification we take the corresponding derived (p, I) -completion.

Definition 3.2.5. Let (A, I) be a perfectoid prism, and we consider any E_∞ -ring \mathcal{O} in the following

$$\text{Object}_{E_\infty\text{-commutativealgebra, Smplicial}}(\text{IndSNorm}_{A/I})^{\text{smoothformalseriesclosure}}, \quad (3.2.84)$$

$$\text{Object}_{E_\infty\text{-commutativealgebra, Smplicial}}(\text{Ind}^m\text{SNorm}_{A/I})^{\text{smoothformalseriesclosure}}, \quad (3.2.85)$$

$$\text{Object}_{E_\infty\text{-commutativealgebra, Smplicial}}(\text{IndNorm}_{A/I})^{\text{smoothformalseriesclosure}}, \quad (3.2.86)$$

$$\text{Object}_{E_\infty\text{-commutativealgebra, Smplicial}}(\text{Ind}^m\text{Norm}_{A/I})^{\text{smoothformalseriesclosure}}, \quad (3.2.87)$$

$$\text{Object}_{E_\infty\text{-commutativealgebra, Smplicial}}(\text{IndBan}_{A/I})^{\text{smoothformalseriesclosure}}, \quad (3.2.88)$$

$$\text{Object}_{E_\infty\text{-commutativealgebra, Smplicial}}(\text{Ind}^m\text{Ban}_{A/I})^{\text{smoothformalseriesclosure}}. \quad (3.2.89)$$

Then consider the derived prismatic object:

$$\text{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, KKM, BBM, formalanalytification}}(\mathcal{O}). \quad (3.2.90)$$

Then as in [BS1, Definition 8.2] we have the following preperfectoidization:

$$(\mathcal{O})^{\text{preperfectoidization}} \quad (3.2.91)$$

$$:= \text{Colimit}(\text{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, KKM, BBM, formalanalytification}}(\mathcal{O}) \rightarrow \quad (3.2.92)$$

$$\text{Fro}_*\text{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, KKM, BBM, formalanalytification}}(\mathcal{O}) \quad (3.2.93)$$

$$\rightarrow \text{Fro}_*\text{Fro}_*\text{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, KKM, BBM, formalanalytification}}(\mathcal{O}) \rightarrow \dots)^{\text{BBM, formalanalytification}}, \quad (3.2.94)$$

after taking the formal series ring left Kan extension analytification from [BBM, Section 4.2], which is defined by taking the left Kan extension to all the $(\infty, 1)$ -ring object in the ∞ -derived category of all A -modules from formal series rings over A . Then we define the corresponding perfectoidization:

$$(\mathcal{O})^{\text{perfectoidization}} \quad (3.2.95)$$

$$:= \text{Colimit}(\text{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, KKM, BBM, formalanalytification}}(\mathcal{O}) \longrightarrow \quad (3.2.96)$$

$$\text{Fro}_*\text{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, KKM, BBM, formalanalytification}}(\mathcal{O}) \quad (3.2.97)$$

$$\longrightarrow \text{Fro}_*\text{Fro}_*\text{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, KKM, BBM, formalanalytification}}(\mathcal{O}) \longrightarrow \dots)^{\text{BBM, formalanalytification}} \times A/I. \quad (3.2.98)$$

Furthermore one can take derived (p, I) -completion to achieve the derived (p, I) -completed versions:

$$\mathcal{O}^{\text{preperfectoidization, derivedcomplete}} := (\mathcal{O}^{\text{preperfectoidization}})^\wedge, \quad (3.2.99)$$

$$\mathcal{O}^{\text{perfectoidization, derivedcomplete}} := \mathcal{O}^{\text{preperfectoidization, derivedcomplete}} \times A/I. \quad (3.2.100)$$

$$(3.2.101)$$

These are large $(\infty, 1)$ -commutative algebra objects in the corresponding categories as in the above, attached to also large $(\infty, 1)$ -commutative algebra objects. When we apply this to the corresponding sub- $(\infty, 1)$ -categories of Banach perfectoid objects in [BMS2], [GR], [KL1], [KL2], [Ked1], [Sch3], we will recover the corresponding distinguished elemental deformation processes defined in [BMS2], [GR], [KL1], [KL2], [Ked1], [Sch3].

Remark 3.2.6. One can then define such ring \mathcal{O} to be *preperfectoid* if we have the equivalence:

$$\mathcal{O}^{\text{preperfectoidization}} \xrightarrow{\sim} \mathcal{O}. \quad (3.2.102)$$

One can then define such ring \mathcal{O} to be *perfectoid* if we have the equivalence:

$$\mathcal{O}^{\text{preperfectoidization}} \times A/I \xrightarrow{\sim} \mathcal{O}. \quad (3.2.103)$$

3.3 Functional Analytic Derived Logarithmic Prismatic Cohomology and Functional Analytic Derived Logarithmic Perfectoidizations

3.3.1 Functional Analytic Derived Logarithmic Prismatic Cohomology

We now extend the previous functional analytic discussion to logarithmic context, which is related to the logarithmic context for instance in [O, Chapter 8], [B1, Chapter 5, Chapter 6, Chapter 7] and [Ko1].

Definition 3.3.1. We define the logarithmic version of the $(\infty, 1)$ -categories we considered in the previous section (let $(B, *)$ be any Banach logarithmic ring or \mathbb{F}_1):

$$\text{Object}_{E_\infty\text{commutativealgebra}, \text{Simplicial}}(\text{IndSNorm}_B)^{\text{prelog}}, \quad (3.3.1)$$

$$\text{Object}_{E_\infty\text{commutativealgebra}, \text{Simplicial}}(\text{Ind}^m\text{SNorm}_B)^{\text{prelog}}, \quad (3.3.2)$$

$$\text{Object}_{E_\infty\text{commutativealgebra}, \text{Simplicial}}(\text{IndNorm}_B)^{\text{prelog}}, \quad (3.3.3)$$

$$\text{Object}_{E_\infty\text{commutativealgebra}, \text{Simplicial}}(\text{Ind}^m\text{Norm}_B)^{\text{prelog}}, \quad (3.3.4)$$

$$\text{Object}_{E_\infty\text{commutativealgebra}, \text{Simplicial}}(\text{IndBan}_B)^{\text{prelog}}, \quad (3.3.5)$$

$$\text{Object}_{E_\infty\text{commutativealgebra}, \text{Simplicial}}(\text{Ind}^m\text{Ban}_B)^{\text{prelog}}, \quad (3.3.6)$$

where the prelog object means a morphism $M \rightarrow ?$ where M is in the following categories of $(\infty, 1)$ -monoid objects:

$$\text{Object}_{E_\infty\text{monoid}, \text{Simplicial}}(\text{IndSNorm}_B), \quad (3.3.7)$$

$$\text{Object}_{E_\infty\text{monoid}, \text{Simplicial}}(\text{Ind}^m\text{SNorm}_B), \quad (3.3.8)$$

$$\text{Object}_{E_\infty\text{monoid}, \text{Simplicial}}(\text{IndNorm}_B), \quad (3.3.9)$$

$$\text{Object}_{E_\infty\text{monoid}, \text{Simplicial}}(\text{Ind}^m\text{Norm}_B), \quad (3.3.10)$$

$$\text{Object}_{E_\infty\text{monoid}, \text{Simplicial}}(\text{IndBan}_B), \quad (3.3.11)$$

$$\text{Object}_{E_\infty\text{monoid}, \text{Simplicial}}(\text{Ind}^m\text{Ban}_B), \quad (3.3.12)$$

$$(3.3.13)$$

with $?$ being commutative algebra object in the previous section.

Assumption 3.3.2. We are going to consider the possible colimits in the $(\infty, 1)$ -categories:

$$\text{Object}_{E_\infty\text{commutativealgebra}, \text{Simplicial}}(\text{IndSNorm}_B)^{\text{prelog}}, \quad (3.3.14)$$

$$\text{Object}_{E_\infty\text{commutativealgebra}, \text{Simplicial}}(\text{Ind}^m\text{SNorm}_B)^{\text{prelog}}, \quad (3.3.15)$$

$$\text{Object}_{E_\infty\text{commutativealgebra}, \text{Simplicial}}(\text{IndNorm}_B)^{\text{prelog}}, \quad (3.3.16)$$

$$\text{Object}_{E_\infty\text{commutativealgebra}, \text{Simplicial}}(\text{Ind}^m\text{Norm}_B)^{\text{prelog}}, \quad (3.3.17)$$

$$\text{Object}_{E_\infty\text{commutativealgebra}, \text{Simplicial}}(\text{IndBan}_B)^{\text{prelog}}, \quad (3.3.18)$$

$$\text{Object}_{E_\infty\text{commutativealgebra}, \text{Simplicial}}(\text{Ind}^m\text{Ban}_B)^{\text{prelog}}, \quad (3.3.19)$$

which means essentially that all the construction will only apply to the object $\mathcal{M} \rightarrow \mathcal{O}$ which could be written as the colimit of logarithmic formal series rings over $(B, *)$:

$$\text{homotopycolimit}_{i \text{ sifted}} (\mathcal{M}_i \rightarrow \mathcal{O}_i)$$

in certain large enough ∞ -category:

$$\text{Object}_{\text{Simplicial}}(\text{IndSets}) \times \text{Object}_{\text{Simplicial}}(\text{IndSets}). \quad (3.3.20)$$

$$(3.3.21)$$

We conjecture the corresponding homotopy colimits exist throughout in order to drop this technical assumption. The resulting ∞ -categories are denoted to be:

$$\text{Object}_{\text{E}_\infty\text{commutativealgebra, Simplicial}}(\text{IndSNorm}_B)^{\text{smoothformalseriesclosure, prelog}}, \quad (3.3.22)$$

$$\text{Object}_{\text{E}_\infty\text{commutativealgebra, Simplicial}}(\text{Ind}^m\text{SNorm}_B)^{\text{smoothformalseriesclosure, prelog}}, \quad (3.3.23)$$

$$\text{Object}_{\text{E}_\infty\text{commutativealgebra, Simplicial}}(\text{IndNorm}_B)^{\text{smoothformalseriesclosure, prelog}}, \quad (3.3.24)$$

$$\text{Object}_{\text{E}_\infty\text{commutativealgebra, Simplicial}}(\text{Ind}^m\text{Norm}_B)^{\text{smoothformalseriesclosure, prelog}}, \quad (3.3.25)$$

$$\text{Object}_{\text{E}_\infty\text{commutativealgebra, Simplicial}}(\text{IndBan}_B)^{\text{smoothformalseriesclosure, prelog}}, \quad (3.3.26)$$

$$\text{Object}_{\text{E}_\infty\text{commutativealgebra, Simplicial}}(\text{Ind}^m\text{Ban}_B)^{\text{smoothformalseriesclosure, prelog}}. \quad (3.3.27)$$

Namely one considers all the logarithmic formal series rings, and then takes in

$$\text{Object}_{\text{Simplicial}}(\text{IndSets}) \times \text{Object}_{\text{Simplicial}}(\text{IndSets}) \quad (3.3.28)$$

the closure by colimits.

Definition 3.3.3. Then we can in the same fashion consider the corresponding derived prismatic complexes [BS1, Construction 7.6], [Ko1, Definition 4.1]⁸ for the commutative algebras as in the above (for a given log prism (A, I, M)):

$$\text{Kan}_{\text{Left}}\Delta_{?/A}, \quad (3.3.29)$$

by the regular corresponding left Kan extension techniques on the following $(\infty, 1)$ -compactly generated closures of the corresponding polynomials given over A/I with a chosen log prism (A, I, M) :

$$\text{Object}_{\text{E}_\infty\text{commutativealgebra, Simplicial}}(\text{IndSNorm}_{A/I})^{\text{smoothformalseriesclosure, prelog}}, \quad (3.3.30)$$

$$\text{Object}_{\text{E}_\infty\text{commutativealgebra, Simplicial}}(\text{Ind}^m\text{SNorm}_{A/I})^{\text{smoothformalseriesclosure, prelog}}, \quad (3.3.31)$$

$$\text{Object}_{\text{E}_\infty\text{commutativealgebra, Simplicial}}(\text{IndNorm}_{A/I})^{\text{smoothformalseriesclosure, prelog}}, \quad (3.3.32)$$

$$\text{Object}_{\text{E}_\infty\text{commutativealgebra, Simplicial}}(\text{Ind}^m\text{Norm}_{A/I})^{\text{smoothformalseriesclosure, prelog}}, \quad (3.3.33)$$

$$\text{Object}_{\text{E}_\infty\text{commutativealgebra, Simplicial}}(\text{IndBan}_{A/I})^{\text{smoothformalseriesclosure, prelog}}, \quad (3.3.34)$$

$$\text{Object}_{\text{E}_\infty\text{commutativealgebra, Simplicial}}(\text{Ind}^m\text{Ban}_{A/I})^{\text{smoothformalseriesclosure, prelog}}. \quad (3.3.35)$$

⁸One just applies [BS1, Construction 7.6], [Ko1, Definition 4.1] and then takes the left Kan extensions. We note that the derived logarithmic prismatic cohomologies are considered in [Ko1, Just above Notation in Section 1].

We call the corresponding functors functional analytic derived logarithmic prismatic complexes which we are going to denote that as in the following:

$$\mathbf{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, logarithmic, KKM, BBM, formalanalytification}} \cdot \quad (3.3.36)$$

This would mean the following definition⁹:

$$\mathbf{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, logarithmic, KKM, BBM, formalanalytification}}(\mathcal{O}) \quad (3.3.37)$$

$$:= \left(\underset{i}{\text{homotopycolimit}} \underset{\text{sifted, derivedcategory}_{\infty}(A/I\text{-Module})}{\mathbf{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, logarithmic, KKM}}} \right) \quad (3.3.38)$$

$$(\mathcal{O}_i)^{\wedge} \quad (3.3.39)$$

$$\text{)BBM, formalanalytification} \quad (3.3.40)$$

by writing any object \mathcal{O} as the corresponding colimit

$$\underset{i}{\text{homotopycolimit}} \underset{\text{sifted}}{\mathcal{O}_i}.$$

These are quite large $(\infty, 1)$ -commutative ring objects in the corresponding $(\infty, 1)$ -categories for $(R = A/I, *)$:

$$\text{Object}_{\mathbb{E}_{\infty}\text{commutativealgebra, Simplicial}}(\text{IndSNorm}_R)^{\text{prelog}}, \quad (3.3.41)$$

$$\text{Object}_{\mathbb{E}_{\infty}\text{commutativealgebra, Simplicial}}(\text{Ind}^m\text{SNorm}_R)^{\text{prelog}}, \quad (3.3.42)$$

$$\text{Object}_{\mathbb{E}_{\infty}\text{commutativealgebra, Simplicial}}(\text{IndNorm}_R)^{\text{prelog}}, \quad (3.3.43)$$

$$\text{Object}_{\mathbb{E}_{\infty}\text{commutativealgebra, Simplicial}}(\text{Ind}^m\text{Norm}_R)^{\text{prelog}}, \quad (3.3.44)$$

$$\text{Object}_{\mathbb{E}_{\infty}\text{commutativealgebra, Simplicial}}(\text{IndBan}_R)^{\text{prelog}}, \quad (3.3.45)$$

$$\text{Object}_{\mathbb{E}_{\infty}\text{commutativealgebra, Simplicial}}(\text{Ind}^m\text{Ban}_R)^{\text{prelog}}, \quad (3.3.46)$$

$$(3.3.47)$$

after taking the formal series ring left Kan extension analytification from [BBM, Section 4.2], which is defined by taking the left Kan extension to all the $(\infty, 1)$ -ring objects in the ∞ -derived category of all A -modules from formal series rings over A .

⁹Before the Ben-Bassat-Mukherjee p -adic formal analytification we take the corresponding derived (p, I) -completion.

3.3.2 Functional Analytic Derived Logarithmic Perfectoidizations

Then as in [BS1, Definition 8.2] we consider the corresponding perfectoidization in this analytic setting.

Definition 3.3.4. Let (A, I, M) be a perfectoid logarithmic prism as in [Ko1, Definition 3.3], and we consider any E_∞ -ring \mathcal{O} in the following

$$\text{Object}_{E_\infty\text{-commutativealgebra, Simplicial}}(\text{IndSNorm}_{A/I})^{\text{smoothformalseriesclosure, prelog}}, \quad (3.3.48)$$

$$\text{Object}_{E_\infty\text{-commutativealgebra, Simplicial}}(\text{Ind}^m\text{SNorm}_{A/I})^{\text{smoothformalseriesclosure, prelog}}, \quad (3.3.49)$$

$$\text{Object}_{E_\infty\text{-commutativealgebra, Simplicial}}(\text{IndNorm}_{A/I})^{\text{smoothformalseriesclosure, prelog}}, \quad (3.3.50)$$

$$\text{Object}_{E_\infty\text{-commutativealgebra, Simplicial}}(\text{Ind}^m\text{Norm}_{A/I})^{\text{smoothformalseriesclosure, prelog}}, \quad (3.3.51)$$

$$\text{Object}_{E_\infty\text{-commutativealgebra, Simplicial}}(\text{IndBan}_{A/I})^{\text{smoothformalseriesclosure, prelog}}, \quad (3.3.52)$$

$$\text{Object}_{E_\infty\text{-commutativealgebra, Simplicial}}(\text{Ind}^m\text{Ban}_{A/I})^{\text{smoothformalseriesclosure, prelog}}. \quad (3.3.53)$$

Then consider the derived prismatic object:

$$\text{Kan}_{\text{Left}\Delta^?/A, \text{functionalanalytic, logarithmic, KKM, BBM, formalanalytification}}(\mathcal{O}) \quad (3.3.54)$$

$$:= \left(\underset{i}{\text{homotopycolimit}} \underset{\text{sifted, derivedcategory}_\infty(A/I\text{-Module})}{\text{Kan}_{\text{Left}\Delta^?/A, \text{functionalanalytic, logarithmic, KKM}}} \right) \quad (3.3.55)$$

$$(\mathcal{O}_i)^\wedge \quad (3.3.56)$$

$$)_{\text{BBM, formalanalytification}} \cdot \quad (3.3.57)$$

Then as in [BS1, Definition 8.2] we have the following preperfectoidization:

$$(\mathcal{O})^{\text{preperfectoidization}} \quad (3.3.58)$$

$$:= \text{Colimit}(\text{Kan}_{\text{Left}\Delta^?/A, \text{functionalanalytic, logarithmic, KKM, BBM, formalanalytification}}(\mathcal{O}) \longrightarrow \quad (3.3.59)$$

$$\text{Fro}_*\text{Kan}_{\text{Left}\Delta^?/A, \text{functionalanalytic, logarithmic, KKM, BBM, formalanalytification}}(\mathcal{O}) \quad (3.3.60)$$

$$\longrightarrow \text{Fro}_*\text{Fro}_*\text{Kan}_{\text{Left}\Delta^?/A, \text{functionalanalytic, logarithmic, KKM, BBM, formalanalytification}}(\mathcal{O}) \longrightarrow \dots \quad (3.3.61)$$

$$)_{\text{BBM, formalanalytification}}. \quad (3.3.62)$$

¹⁰Then we define the corresponding perfectoidization:

$$(\mathcal{O})^{\text{perfectoidization}} \quad (3.3.70)$$

$$:= \text{Colimit}(\text{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, logarithmic, KKM, BBM, formalanalytification}}(\mathcal{O}) \longrightarrow \quad (3.3.71)$$

$$\text{Fro}_*\text{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, logarithmic, KKM, BBM, formalanalytification}}(\mathcal{O}) \quad (3.3.72)$$

$$\longrightarrow \text{Fro}_*\text{Fro}_*\text{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, logarithmic, KKM, BBM, formalanalytification}}(\mathcal{O}) \longrightarrow \quad (3.3.73)$$

$$\dots)_{\text{BBM, formalanalytification}} \times A/I. \quad (3.3.74)$$

Furthermore one can take derived (p, I) -completion to achieve the derived (p, I) -completed versions:

$$\mathcal{O}^{\text{preperfectoidization, derivedcomplete}} := (\mathcal{O}^{\text{preperfectoidization}})^\wedge, \quad (3.3.75)$$

$$\mathcal{O}^{\text{perfectoidization, derivedcomplete}} := \mathcal{O}^{\text{preperfectoidization, derivedcomplete}} \times A/I. \quad (3.3.76)$$

$$(3.3.77)$$

These are large $(\infty, 1)$ -commutative algebra objects in the corresponding categories as in the above, attached to also large $(\infty, 1)$ -commutative algebra objects. When we apply this to the corresponding sub- $(\infty, 1)$ -categories of Banach perfectoid objects in [BMS2], [DLLZ1], [DLLZ2], [GR], [KL1], [KL2], [Ked1], [Sch3] we will recover the corresponding distinguished elemental deformation processes defined in [BMS2], [DLLZ1], [DLLZ2], [GR], [KL1], [KL2], [Ked1], [Sch3].

Remark 3.3.5. One can then define such ring \mathcal{O} to be *logarithmicpreperfectoid* if we have the equivalence:

$$\mathcal{O}^{\text{preperfectoidization}} \xrightarrow{\sim} \mathcal{O}. \quad (3.3.78)$$

One can then define such ring \mathcal{O} to be *logarithmicperfectoid* if we have the equivalence:

$$\mathcal{O}^{\text{preperfectoidization}} \times A/I \xrightarrow{\sim} \mathcal{O}. \quad (3.3.79)$$

¹⁰Again after taking the formal series ring left Kan extension analytification from [BBM, Section 4.2], which is defined by taking the left Kan extension to all the $(\infty, 1)$ -ring objects in the ∞ -derived category of all A -modules from formal series rings over A , into:

$$\text{Object}_{\text{E}_\infty \text{commutativealgebra, Simplicial}}(\text{IndSNorm}_R)^{\text{prelog}}, \quad (3.3.63)$$

$$\text{Object}_{\text{E}_\infty \text{commutativealgebra, Simplicial}}(\text{Ind}^m \text{SNorm}_R)^{\text{prelog}}, \quad (3.3.64)$$

$$\text{Object}_{\text{E}_\infty \text{commutativealgebra, Simplicial}}(\text{IndNorm}_R)^{\text{prelog}}, \quad (3.3.65)$$

$$\text{Object}_{\text{E}_\infty \text{commutativealgebra, Simplicial}}(\text{Ind}^m \text{Norm}_R)^{\text{prelog}}, \quad (3.3.66)$$

$$\text{Object}_{\text{E}_\infty \text{commutativealgebra, Simplicial}}(\text{IndBan}_R)^{\text{prelog}}, \quad (3.3.67)$$

$$\text{Object}_{\text{E}_\infty \text{commutativealgebra, Simplicial}}(\text{Ind}^m \text{Ban}_R)^{\text{prelog}}. \quad (3.3.68)$$

$$(3.3.69)$$

3.4 Functional Analytic Derived Prismatic Complexes for $(\infty, 1)$ -Analytic Stacks and the Preperfectoidizations

3.4.1 Functional Analytic Derived Prismatic Complexes for $(\infty, 1)$ -Analytic Stacks

We now promote the construction in the previous sections to the corresponding $(\infty, 1)$ -ringed toposes level after Lurie [Lu1], [Lu2] and [Lu3] in the ∞ -category of ∞ -ringed toposes, Bambozzi-Ben-Bassat-Kremnizer [BBBK], Ben-Bassat-Mukherjee [BBM], Bambozzi-Kremnizer [BK], Clausen-Scholze [CS1] [CS2] and Kelly-Kremnizer-Mukherjee [KKM] in the ∞ -category of ∞ -functional analytic ringed toposes.

Now we consider the following ∞ -categories of the corresponding ∞ -analytic ringed toposes from Bambozzi-Ben-Bassat-Kremnizer [BBBK]:

$$\infty - \text{Toposes}^{\text{ringed, commutative algebra}_{\text{simplicial}}(\text{IndSeminormed}_?)} \text{Commutative algebra}_{\text{simplicial}}(\text{IndSeminormed}_?)^{\text{opposite}}, \text{Grothendieck topology, homotopy epimorphism} \cdot \quad (3.4.1)$$

$$\infty - \text{Toposes}^{\text{ringed, Commutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_?)} \text{Commutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_?)^{\text{opposite}}, \text{Grothendieck topology, homotopy epimorphism} \cdot \quad (3.4.2)$$

$$\infty - \text{Toposes}^{\text{ringed, Commutative algebra}_{\text{simplicial}}(\text{IndNormed}_?)} \text{Commutative algebra}_{\text{simplicial}}(\text{IndNormed}_?)^{\text{opposite}}, \text{Grothendieck topology, homotopy epimorphism} \cdot \quad (3.4.3)$$

$$\infty - \text{Toposes}^{\text{ringed, Commutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_?)} \text{Commutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_?)^{\text{opposite}}, \text{Grothendieck topology, homotopy epimorphism} \cdot \quad (3.4.4)$$

$$\infty - \text{Toposes}^{\text{ringed, Commutative algebra}_{\text{simplicial}}(\text{IndBanach}_?)} \text{Commutative algebra}_{\text{simplicial}}(\text{IndBanach}_?)^{\text{opposite}}, \text{Grothendieck topology, homotopy epimorphism} \cdot \quad (3.4.5)$$

$$\infty - \text{Toposes}^{\text{ringed, Commutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_?)} \text{Commutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_?)^{\text{opposite}}, \text{Grothendieck topology, homotopy epimorphism} \cdot \quad (3.4.6)$$

$$\text{Proj}^{\text{smooth formal series closure}} \infty - \text{Toposes}^{\text{ringed, commutative algebra}_{\text{simplicial}}(\text{IndSeminormed}_?)} \text{Commutative algebra}_{\text{simplicial}}(\text{IndSeminormed}_?)^{\text{opposite}}, \text{Grothendieck topology, homotopy epimorphism} \cdot \quad (3.4.7)$$

$$\text{Proj}^{\text{smooth formal series closure}} \infty - \text{Toposes}^{\text{ringed, Commutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_?)} \text{Commutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_?)^{\text{opposite}}, \text{Grothendieck topology, homotopy epimorphism} \cdot \quad (3.4.8)$$

$$\text{Proj}^{\text{smooth formal series closure}} \infty - \text{Toposes}^{\text{ringed, Commutative algebra}_{\text{simplicial}}(\text{IndNormed}_?)} \text{Commutative algebra}_{\text{simplicial}}(\text{IndNormed}_?)^{\text{opposite}}, \text{Grothendieck topology, homotopy epimorphism} \cdot \quad (3.4.9)$$

$$\text{Proj}^{\text{smooth formal series closure}} \infty - \text{Toposes}^{\text{ringed, Commutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_?)} \text{Commutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_?)^{\text{opposite}}, \text{Grothendieck topology, homotopy epimorphism} \cdot \quad (3.4.10)$$

$$\text{Proj}^{\text{smooth formal series closure}} \infty - \text{Toposes}^{\text{ringed, Commutative algebra}_{\text{simplicial}}(\text{IndBanach}_?)} \text{Commutative algebra}_{\text{simplicial}}(\text{IndBanach}_?)^{\text{opposite}}, \text{Grothendieck topology, homotopy epimorphism} \cdot \quad (3.4.11)$$

$$\text{Proj}^{\text{smooth formal series closure}} \infty - \text{Toposes}^{\text{ringed, Commutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_?)} \text{Commutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_?)^{\text{opposite}}, \text{Grothendieck topology, homotopy epimorphism} \cdot \quad (3.4.12)$$

$$(3.4.13)$$

Example 3.4.1. (Preadic nonsheafy spaces and their colimits) The main example we would like consider comes from [BK] where one constructs the corresponding derived adic space ($\text{Spectrum}_{\text{adic}}_{\text{BK}}(R), \mathcal{O}_{\text{Spec}}$) by using derived rational localization to reach some ∞ -toposes carrying essentially ∞ -sheaf of rings, from any Banach ring R^{11} . Then we apply this to the corresponding situation below. Let (A, I) be a corresponding prism from Bhatt-Scholze, with the assumption that A/I is Banach. Then what we do is consider all the formal series rings over A/I then take the colimit completion in the homotopy sense, by embedding them through $\text{Spectrum}_{\text{adic}}_{\text{BK}}$ into the ∞ -toposes

¹¹With some key essential assumptions but that is not so serious once one considers some foundation from Kedlaya in [Ked2] on reified adic spaces as mentioned in the last section of [BK].

ringed. For instance for any such formal ring F , one could regard this as a inductive system of Banach rings:

$$F = \operatorname{homotopycolimit}_i F_i, \quad (3.4.14)$$

where we set:

$$(\operatorname{Spectrumadic}_{\text{BK}}(F), \mathcal{O}) := \operatorname{homotopylimit}_i (\operatorname{Spectrumadic}_{\text{BK}}(F_i), \mathcal{O}_{\operatorname{Spectrumadic}_{\text{BK}}(F_i)}). \quad (3.4.15)$$

The resulting ∞ -stacks generated are interesting to study.

Definition 3.4.2. One can actually define the derived prismatic cohomology presheaves through derived topological Hochschild cohomology presheaves, derived topological period cohomology presheaves and derived topological cyclic cohomology presheaves as in [BMS, Section 2.2, Section 2.3], [BS1, Theorem 1.13]:

$$\operatorname{Kan}_{\text{Left}} \operatorname{THH}, \operatorname{Kan}_{\text{Left}} \operatorname{TP}, \operatorname{Kan}_{\text{Left}} \operatorname{TC}, \quad (3.4.16)$$

on the following $(\infty, 1)$ -compactly generated closures of the corresponding polynomials¹² given over A/I with a chosen prism (A, I) ¹³:

$$\operatorname{Proj}^{\operatorname{smoothformalseriesclosure}}_{\infty} - \operatorname{Toposes}_{\substack{\text{ringed, commutative algebra}_{\text{simplicial}}(\operatorname{IndSeminormed}_{A/I}) \\ \text{Commutative algebra}_{\text{simplicial}}(\operatorname{IndSeminormed}_{A/I})^{\text{opposite}}, \text{Grothendieck topology, homotopyepimorphisms}}} \quad (3.4.17)$$

$$\operatorname{Proj}^{\operatorname{smoothformalseriesclosure}}_{\infty} - \operatorname{Toposes}_{\substack{\text{ringed, Commutative algebra}_{\text{simplicial}}(\operatorname{Ind}^m \operatorname{Seminormed}_{A/I}) \\ \text{Commutative algebra}_{\text{simplicial}}(\operatorname{Ind}^m \operatorname{Seminormed}_{A/I})^{\text{opposite}}, \text{Grothendieck topology, homotopyepimorphisms}}} \quad (3.4.18)$$

$$\operatorname{Proj}^{\operatorname{smoothformalseriesclosure}}_{\infty} - \operatorname{Toposes}_{\substack{\text{ringed, Commutative algebra}_{\text{simplicial}}(\operatorname{IndNormed}_{A/I}) \\ \text{Commutative algebra}_{\text{simplicial}}(\operatorname{IndNormed}_{A/I})^{\text{opposite}}, \text{Grothendieck topology, homotopyepimorphisms}}} \quad (3.4.19)$$

$$\operatorname{Proj}^{\operatorname{smoothformalseriesclosure}}_{\infty} - \operatorname{Toposes}_{\substack{\text{ringed, Commutative algebra}_{\text{simplicial}}(\operatorname{Ind}^m \operatorname{Normed}_{A/I}) \\ \text{Commutative algebra}_{\text{simplicial}}(\operatorname{Ind}^m \operatorname{Normed}_{A/I})^{\text{opposite}}, \text{Grothendieck topology, homotopyepimorphisms}}} \quad (3.4.20)$$

$$\operatorname{Proj}^{\operatorname{smoothformalseriesclosure}}_{\infty} - \operatorname{Toposes}_{\substack{\text{ringed, Commutative algebra}_{\text{simplicial}}(\operatorname{IndBanach}_{A/I}) \\ \text{Commutative algebra}_{\text{simplicial}}(\operatorname{IndBanach}_{A/I})^{\text{opposite}}, \text{Grothendieck topology, homotopyepimorphisms}}} \quad (3.4.21)$$

$$\operatorname{Proj}^{\operatorname{smoothformalseriesclosure}}_{\infty} - \operatorname{Toposes}_{\substack{\text{ringed, Commutative algebra}_{\text{simplicial}}(\operatorname{Ind}^m \operatorname{Banach}_{A/I}) \\ \text{Commutative algebra}_{\text{simplicial}}(\operatorname{Ind}^m \operatorname{Banach}_{A/I})^{\text{opposite}}, \text{Grothendieck topology, homotopyepimorphisms}}} \quad (3.4.22)$$

We call the corresponding functors are derived functional analytic Hochschild cohomology presheaves, derived functional analytic period cohomology presheaves and derived functional analytic cyclic cohomology presheaves, which we are going to denote these presheaves as in

¹²Definitely, we need to put certain norms over in some relatively canonical way, as in [BBM, Section 4.2] one can basically consider rigid ones and dagger ones, and so on. We restrict to the *formal* one.

¹³In all the following, we assume this prism to be bounded and satisfy that A/I is Banach.

the following for any ∞ -ringed topos $(\mathbb{X}, \mathcal{O}) = \mathop{\mathrm{homotopycolimit}}_i(\mathbb{X}_i, \mathcal{O}_i)$:

$$\mathop{\mathrm{Kan}}_{\mathrm{Left}} \mathop{\mathrm{THH}}_{\mathrm{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \mathrm{formalanalytification}}(\mathcal{O}) := \quad (3.4.23)$$

$$\left(\mathop{\mathrm{homotopycolimit}}_i \quad \mathop{\mathrm{Kan}}_{\mathrm{Left}} \mathop{\mathrm{THH}}_{\mathrm{functionalanalytic}, \mathrm{KKM}}(\mathcal{O}_i) \right)_{\mathrm{sifted}, \mathrm{derivedcategory}_\infty(A/I\text{-Module})} \quad (3.4.24)$$

$$\left. \right)_{\mathrm{BBM}, \mathrm{formalanalytification}} \quad (3.4.25)$$

$$\mathop{\mathrm{Kan}}_{\mathrm{Left}} \mathop{\mathrm{TP}}_{\mathrm{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \mathrm{formalanalytification}}(\mathcal{O}) := \quad (3.4.26)$$

$$\left(\mathop{\mathrm{homotopycolimit}}_i \quad \mathop{\mathrm{Kan}}_{\mathrm{Left}} \mathop{\mathrm{TP}}_{\mathrm{functionalanalytic}, \mathrm{KKM}}(\mathcal{O}_i) \right)_{\mathrm{sifted}, \mathrm{derivedcategory}_\infty(A/I\text{-Module})} \quad (3.4.27)$$

$$\left. \right)_{\mathrm{BBM}, \mathrm{formalanalytification}} \quad (3.4.28)$$

$$\mathop{\mathrm{Kan}}_{\mathrm{Left}} \mathop{\mathrm{TC}}_{\mathrm{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \mathrm{formalanalytification}}(\mathcal{O}) := \quad (3.4.29)$$

$$\left(\mathop{\mathrm{homotopycolimit}}_i \quad \mathop{\mathrm{Kan}}_{\mathrm{Left}} \mathop{\mathrm{TC}}_{\mathrm{functionalanalytic}, \mathrm{KKM}}(\mathcal{O}_i) \right)_{\mathrm{sifted}, \mathrm{derivedcategory}_\infty(A/I\text{-Module})} \quad (3.4.30)$$

$$\left. \right)_{\mathrm{BBM}, \mathrm{formalanalytification}} \quad (3.4.31)$$

by writing any object \mathcal{O} as the corresponding colimit

$$\mathop{\mathrm{homotopycolimit}}_i \quad \mathcal{O}_i := \mathop{\mathrm{homotopycolimit}}_i \quad \mathcal{O}_{\mathrm{Spectrumadic}_{\mathrm{BK}}(\mathrm{Formalseriesring}_i)}$$

These are quite large $(\infty, 1)$ -commutative ring objects in the corresponding $(\infty, 1)$ -categories for $? = A/I$ over $(\mathbb{X}, \mathcal{O})$:

$$\mathop{\mathrm{Ind}}_{\mathrm{smoothformalseriesclosure}_\infty\text{-Toposes}} \# \mathop{\mathrm{Quasicoherent}}^{\mathrm{presheaf}}_{\mathrm{ringed}, \mathrm{commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndSeminormed}_?)^{\mathrm{opposite}}, \mathrm{Grotopology}, \mathrm{homotopyepimorphism}} \quad (3.4.32)$$

$$\mathop{\mathrm{Ind}}_{\mathrm{smoothformalseriesclosure}_\infty\text{-Toposes}} \# \mathop{\mathrm{Quasicoherent}}^{\mathrm{presheaf}}_{\mathrm{ringed}, \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Seminormed}_?)^{\mathrm{opposite}}, \mathrm{Grotopology}, \mathrm{homotopyepimorphism}} \quad (3.4.33)$$

$$\mathop{\mathrm{Ind}}_{\mathrm{smoothformalseriesclosure}_\infty\text{-Toposes}} \# \mathop{\mathrm{Quasicoherent}}^{\mathrm{presheaf}}_{\mathrm{ringed}, \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndNormed}_?)^{\mathrm{opposite}}, \mathrm{Grotopology}, \mathrm{homotopyepimorphism}} \quad (3.4.34)$$

$$\mathop{\mathrm{Ind}}_{\mathrm{smoothformalseriesclosure}_\infty\text{-Toposes}} \# \mathop{\mathrm{Quasicoherent}}^{\mathrm{presheaf}}_{\mathrm{ringed}, \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Normed}_?)^{\mathrm{opposite}}, \mathrm{Grotopology}, \mathrm{homotopyepimorphism}} \quad (3.4.35)$$

$$\mathop{\mathrm{Ind}}_{\mathrm{smoothformalseriesclosure}_\infty\text{-Toposes}} \# \mathop{\mathrm{Quasicoherent}}^{\mathrm{presheaf}}_{\mathrm{ringed}, \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndBanach}_?)^{\mathrm{opposite}}, \mathrm{Grotopology}, \mathrm{homotopyepimorphism}} \quad (3.4.36)$$

$$\mathop{\mathrm{Ind}}_{\mathrm{smoothformalseriesclosure}_\infty\text{-Toposes}} \# \mathop{\mathrm{Quasicoherent}}^{\mathrm{presheaf}}_{\mathrm{ringed}, \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Banach}_?)^{\mathrm{opposite}}, \mathrm{Grotopology}, \mathrm{homotopyepimorphism}} \quad (3.4.37)$$

$$(3.4.38)$$

after taking the formal series ring left Kan extension analytification from [BBM, Section 4.2], which is defined by taking the left Kan extension to all the $(\infty, 1)$ -ring objects in the ∞ -derived category of all A -modules from formal series rings over A .

Definition 3.4.3. Then we can in the same fashion consider the corresponding derived prismatic complex presheaves [BS1, Construction 7.6]¹⁴ for the commutative algebras as in the above (for a given prism (A, I)):

$$\mathrm{Kan}_{\mathrm{Left}}\Delta_{?/A}, \quad (3.4.39)$$

by the regular corresponding left Kan extension techniques on the following $(\infty, 1)$ -compactly generated closures of the corresponding polynomials given over A/I with a chosen prism (A, I) :

$$\mathrm{Proj}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, commutative algebra}_{\mathrm{simplicial}}(\mathrm{IndSeminormed}_{A/I})} \mathrm{Commutative algebra}_{\mathrm{simplicial}}(\mathrm{IndSeminormed}_{A/I})^{\mathrm{opposite}, \mathrm{Grothendieck topology, homotopyepimorphisms}} \quad (3.4.40)$$

$$\mathrm{Proj}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, Commutative algebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Seminormed}_{A/I})} \mathrm{Commutative algebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Seminormed}_{A/I})^{\mathrm{opposite}, \mathrm{Grothendieck topology, homotopyepimorphisms}} \quad (3.4.41)$$

$$\mathrm{Proj}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, Commutative algebra}_{\mathrm{simplicial}}(\mathrm{IndNormed}_{A/I})} \mathrm{Commutative algebra}_{\mathrm{simplicial}}(\mathrm{IndNormed}_{A/I})^{\mathrm{opposite}, \mathrm{Grothendieck topology, homotopyepimorphisms}} \quad (3.4.42)$$

$$\mathrm{Proj}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, Commutative algebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Normed}_{A/I})} \mathrm{Commutative algebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Normed}_{A/I})^{\mathrm{opposite}, \mathrm{Grothendieck topology, homotopyepimorphisms}} \quad (3.4.43)$$

$$\mathrm{Proj}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, Commutative algebra}_{\mathrm{simplicial}}(\mathrm{IndBanach}_{A/I})} \mathrm{Commutative algebra}_{\mathrm{simplicial}}(\mathrm{IndBanach}_{A/I})^{\mathrm{opposite}, \mathrm{Grothendieck topology, homotopyepimorphisms}} \quad (3.4.44)$$

$$\mathrm{Proj}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, Commutative algebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Banach}_{A/I})} \mathrm{Commutative algebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Banach}_{A/I})^{\mathrm{opposite}, \mathrm{Grothendieck topology, homotopyepimorphisms}} \quad (3.4.45)$$

We call the corresponding functors functional analytic derived prismatic complex presheaves which we are going to denote that as in the following:

$$\mathrm{Kan}_{\mathrm{Left}}\Delta_{?/A, \mathrm{functional analytic, KKM, BBM, formal analytification}} \quad (3.4.46)$$

¹⁴One just applies [BS1, Construction 7.6] and then takes the left Kan extensions.

This would mean the following definition¹⁵:

$$\text{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, KKM, BBM, formalanalytification}}(\mathcal{O}) \quad (3.4.47)$$

$$:= \left(\underset{i}{\text{homotopycolimit}} \underset{\text{sifted, derivedcategory}_{\infty}(A/I\text{-Module})}{\text{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, KKM}}(\mathcal{O}_i) \wedge} \right) \quad (3.4.48)$$

$$\text{)BBM, formalanalytification} \quad (3.4.49)$$

by writing any object \mathcal{O} as the corresponding colimit¹⁶

$$\underset{i}{\text{homotopycolimit}} \underset{\text{sifted}}{\mathcal{O}_i}.$$

These are quite large $(\infty, 1)$ -commutative ring objects in the corresponding $(\infty, 1)$ -categories for $R = A/I$:

$$\text{Ind}\# \text{Quasicoherent}^{\text{presheaf}} \underset{\text{Ind}^{\text{smoothformalseriesclosure}}_{\infty}\text{-Toposes}}{\text{ringed, commutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_R)} \underset{\text{Commutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}}{\cdot} \quad (3.4.53)$$

$$\text{Ind}\# \text{Quasicoherent}^{\text{presheaf}} \underset{\text{Ind}^{\text{smoothformalseriesclosure}}_{\infty}\text{-Toposes}}{\text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_R)} \underset{\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}}{\cdot} \quad (3.4.54)$$

$$\text{Ind}\# \text{Quasicoherent}^{\text{presheaf}} \underset{\text{Ind}^{\text{smoothformalseriesclosure}}_{\infty}\text{-Toposes}}{\text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{IndNormed}_R)} \underset{\text{Commutativealgebra}_{\text{simplicial}}(\text{IndNormed}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}}{\cdot} \quad (3.4.55)$$

$$\text{Ind}\# \text{Quasicoherent}^{\text{presheaf}} \underset{\text{Ind}^{\text{smoothformalseriesclosure}}_{\infty}\text{-Toposes}}{\text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_R)} \underset{\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}}{\cdot} \quad (3.4.56)$$

$$\text{Ind}\# \text{Quasicoherent}^{\text{presheaf}} \underset{\text{Ind}^{\text{smoothformalseriesclosure}}_{\infty}\text{-Toposes}}{\text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{IndBanach}_R)} \underset{\text{Commutativealgebra}_{\text{simplicial}}(\text{IndBanach}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}}{\cdot} \quad (3.4.57)$$

$$\text{Ind}\# \text{Quasicoherent}^{\text{presheaf}} \underset{\text{Ind}^{\text{smoothformalseriesclosure}}_{\infty}\text{-Toposes}}{\text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_R)} \underset{\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}}{\cdot} \quad (3.4.58)$$

$$(3.4.59)$$

¹⁵Before the Ben-Bassat-Mukherjee p -adic formal analytification we take the corresponding derived (p, I) -completion.

¹⁶Here we remind the readers of the corresponding foundation here, namely the presheaf \mathcal{O}_i in fact takes the value in Koszul complex taking the following form:

$$\text{Koszulcomplex}_{A/I\langle X_1, \dots, X_l \rangle \langle T_1, \dots, T_m \rangle}(a_1 - T_1 b_1, \dots, a_m - T_m b_m) \quad (3.4.50)$$

$$= A/I\langle X_1, \dots, X_l \rangle \langle T_1, \dots, T_m \rangle / \mathbb{L}(a_1 - T_1 b_1, \dots, a_m - T_m b_m). \quad (3.4.51)$$

This is actually derived p -complete since each homotopy group is derived p -complete (from the corresponding Banach structure from $A/I\langle X_1, \dots, X_l \rangle$ induced from the p -adic topology). Namely the definition of the presheaf:

$$\text{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, KKM}}(\mathcal{O}_i) \quad (3.4.52)$$

is directly the application of the derived prismatic functor from [BS1, Construction 7.6].

after taking the formal series ring left Kan extension analytification from [BBM, Section 4.2], which is defined by taking the left Kan extension to all the $(\infty, 1)$ -ring objects in the ∞ -derived category of all A -modules from formal series rings over A .

3.4.2 Functional Analytic Derived Preperfectoidizations for $(\infty, 1)$ -Analytic Stacks

Then as in [BS1, Definition 8.2] we consider the corresponding perfectoidization in this analytic setting.

Definition 3.4.4. Let (A, I) be a perfectoid prism, and we consider any E_∞ -ring \mathcal{O} in the following

$$\mathrm{Proj}^{\mathrm{smoothformalseriesclosure}}_\infty - \mathrm{Toposes}_{\mathrm{ringed, commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndSeminormed}_{A/I}) / \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndSeminormed}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology, homotopyepimorphisms}} \quad (3.4.60)$$

$$\mathrm{Proj}^{\mathrm{smoothformalseriesclosure}}_\infty - \mathrm{Toposes}_{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Seminormed}_{A/I}) / \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Seminormed}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology, homotopyepimorphisms}} \quad (3.4.61)$$

$$\mathrm{Proj}^{\mathrm{smoothformalseriesclosure}}_\infty - \mathrm{Toposes}_{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndNormed}_{A/I}) / \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndNormed}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology, homotopyepimorphisms}} \quad (3.4.62)$$

$$\mathrm{Proj}^{\mathrm{smoothformalseriesclosure}}_\infty - \mathrm{Toposes}_{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Normed}_{A/I}) / \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Normed}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology, homotopyepimorphisms}} \quad (3.4.63)$$

$$\mathrm{Proj}^{\mathrm{smoothformalseriesclosure}}_\infty - \mathrm{Toposes}_{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndBanach}_{A/I}) / \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndBanach}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology, homotopyepimorphisms}} \quad (3.4.64)$$

$$\mathrm{Proj}^{\mathrm{smoothformalseriesclosure}}_\infty - \mathrm{Toposes}_{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Banach}_{A/I}) / \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Banach}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology, homotopyepimorphisms}} \quad (3.4.65)$$

Then consider the derived prismatic object:

$$\mathrm{Kan}_{\mathrm{Left}} \Delta_{?, A, \mathrm{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \mathrm{formalanalytification}}(\mathcal{O}). \quad (3.4.66)$$

Then as in [BS1, Definition 8.2] we have the following preperfectoidization:

$$(\mathcal{O})^{\mathrm{preperfectoidization}} \quad (3.4.67)$$

$$:= \mathrm{Colimit}(\mathrm{Kan}_{\mathrm{Left}} \Delta_{?, A, \mathrm{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \mathrm{formalanalytification}}(\mathcal{O}) \rightarrow \dots) \quad (3.4.68)$$

$$\mathrm{Fro}_* \mathrm{Kan}_{\mathrm{Left}} \Delta_{?, A, \mathrm{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \mathrm{formalanalytification}}(\mathcal{O}) \quad (3.4.69)$$

$$\rightarrow \mathrm{Fro}_* \mathrm{Fro}_* \mathrm{Kan}_{\mathrm{Left}} \Delta_{?, A, \mathrm{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \mathrm{formalanalytification}}(\mathcal{O}) \rightarrow \dots)^{\mathrm{BBM}, \mathrm{formalanalytification}}, \quad (3.4.70)$$

after taking the formal series ring left Kan extension analytification from [BBM, Section 4.2], which is defined by taking the left Kan extension to all the $(\infty, 1)$ -ring object in the ∞ -derived category of all A -modules from formal series rings over A . Then we define the corresponding perfectoidization:

$$(\mathcal{O})^{\mathrm{perfectoidization}} \quad (3.4.71)$$

$$:= \mathrm{Colimit}(\mathrm{Kan}_{\mathrm{Left}} \Delta_{?, A, \mathrm{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \mathrm{formalanalytification}}(\mathcal{O}) \rightarrow \dots) \quad (3.4.72)$$

$$\mathrm{Fro}_* \mathrm{Kan}_{\mathrm{Left}} \Delta_{?, A, \mathrm{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \mathrm{formalanalytification}}(\mathcal{O}) \quad (3.4.73)$$

$$\rightarrow \mathrm{Fro}_* \mathrm{Fro}_* \mathrm{Kan}_{\mathrm{Left}} \Delta_{?, A, \mathrm{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \mathrm{formalanalytification}}(\mathcal{O}) \rightarrow \dots)^{\mathrm{BBM}, \mathrm{formalanalytification}} \times A/I. \quad (3.4.74)$$

Furthermore one can take derived (p, I) -completion to achieve the derived (p, I) -completed versions:

$$\mathcal{O}^{\text{preperfectoidization, derived complete}} := (\mathcal{O}^{\text{preperfectoidization}})^{\wedge}, \quad (3.4.75)$$

$$\mathcal{O}^{\text{perfectoidization, derived complete}} := \mathcal{O}^{\text{preperfectoidization, derived complete}} \times A/I. \quad (3.4.76)$$

$$(3.4.77)$$

These are large $(\infty, 1)$ -commutative algebra objects in the corresponding categories as in the above, attached to also large $(\infty, 1)$ -commutative algebra objects. When we apply this to the corresponding sub- $(\infty, 1)$ -categories of Banach perfectoid objects in [BMS2], [GR], [KL1], [KL2], [Ked1], [Sch3], we will recover the corresponding distinguished elemental deformation processes defined in [BMS2], [GR], [KL1], [KL2], [Ked1], [Sch3].

The ∞ -presheaves in this section in the ∞ -category:

$$\text{Ind}\#_{\text{Quasicoherent}}^{\text{presheaf}} \text{Ind}^{\text{smoothformalseriesclosure}} \infty\text{-Toposes} \begin{array}{l} \text{ringed, commutative algebra}_{\text{simplicial}}(\text{IndSeminormed}_R) \\ \text{Commutative algebra}_{\text{simplicial}}(\text{IndSeminormed}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism} \end{array} \cdot \quad (3.4.78)$$

$$\text{Ind}\#_{\text{Quasicoherent}}^{\text{presheaf}} \text{Ind}^{\text{smoothformalseriesclosure}} \infty\text{-Toposes} \begin{array}{l} \text{ringed, Commutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_R) \\ \text{Commutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism} \end{array} \cdot \quad (3.4.79)$$

$$\text{Ind}\#_{\text{Quasicoherent}}^{\text{presheaf}} \text{Ind}^{\text{smoothformalseriesclosure}} \infty\text{-Toposes} \begin{array}{l} \text{ringed, Commutative algebra}_{\text{simplicial}}(\text{IndNormed}_R) \\ \text{Commutative algebra}_{\text{simplicial}}(\text{IndNormed}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism} \end{array} \cdot \quad (3.4.80)$$

$$\text{Ind}\#_{\text{Quasicoherent}}^{\text{presheaf}} \text{Ind}^{\text{smoothformalseriesclosure}} \infty\text{-Toposes} \begin{array}{l} \text{ringed, Commutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_R) \\ \text{Commutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism} \end{array} \cdot \quad (3.4.81)$$

$$\text{Ind}\#_{\text{Quasicoherent}}^{\text{presheaf}} \text{Ind}^{\text{smoothformalseriesclosure}} \infty\text{-Toposes} \begin{array}{l} \text{ringed, Commutative algebra}_{\text{simplicial}}(\text{IndBanach}_R) \\ \text{Commutative algebra}_{\text{simplicial}}(\text{IndBanach}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism} \end{array} \cdot \quad (3.4.82)$$

$$\text{Ind}\#_{\text{Quasicoherent}}^{\text{presheaf}} \text{Ind}^{\text{smoothformalseriesclosure}} \infty\text{-Toposes} \begin{array}{l} \text{ringed, Commutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_R) \\ \text{Commutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism} \end{array} \cdot \quad (3.4.83)$$

$$(3.4.84)$$

are expected to be ∞ -sheaves as long as one considers in the admissible situations the corresponding Čech ∞ -descent for general seminormed modules as in [KKM, Section 9.3] and [BBK] such as Bambozzi-Kremnizer spaces in [BK]. Therefore we have:

Proposition 3.4.5. *The motivic complex ∞ -presheaf*

$$\text{Kan}_{\text{Left}} \Delta? / A, \text{functionalanalytic, KKM, BBM, formalanalytification}(\mathcal{O}), \quad (3.4.85)$$

$$(3.4.86)$$

as well as the corresponding Hodge-Tate ∞ -presheaf as in [BS1]¹⁷:

$$\mathrm{Kan}_{\mathrm{Left}}\Delta_{?/A, \text{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \text{formalanalytification}}(\mathcal{O})^{\mathrm{HodgeTate}} \quad (3.4.87)$$

$$:= \mathrm{Kan}_{\mathrm{Left}}\Delta_{?/A, \text{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \text{formalanalytification}}(\mathcal{O}) \times A/I \quad (3.4.88)$$

$$= \left(\left(\underset{i}{\text{homotopycolimit}} \underset{\text{sifted, derivedcategory}_{\infty}(A/I\text{-Module})}{\mathrm{Kan}_{\mathrm{Left}}\bar{\Delta}_{?/A, \text{functionalanalytic}, \mathrm{KKM}}(\mathcal{O}_i)} \right)^{\wedge} \right) \quad (3.4.89)$$

$$\left. \right)_{\mathrm{BBM}, \text{formalanalytification}} \quad (3.4.90)$$

and the preperfectoidization ∞ -presheaves:

$$(\mathcal{O})^{\text{preperfectoidization}}, \quad (3.4.91)$$

$$(\mathcal{O})^{\text{perfectoidization}}, \quad (3.4.92)$$

$$(\mathcal{O})^{\text{preperfectoidization, derivedcompleted}}, \quad (3.4.93)$$

$$(\mathcal{O})^{\text{perfectoidization, derivedcompleted}} \quad (3.4.94)$$

are ∞ -sheaves over Bambozzi-Kremnizer topos

$$(\mathrm{Spectrum}_{\mathrm{adic}_{\mathrm{BK}}}(F), \mathcal{O}), \quad (3.4.95)$$

attached to any colimit of formal series rings F over A/I in [BK].

Remark 3.4.6. One can then define such ring \mathcal{O} to be *preperfectoid* if we have the equivalence:

$$\mathcal{O}^{\text{preperfectoidization}} \xrightarrow{\sim} \mathcal{O}. \quad (3.4.96)$$

One can then define such ring \mathcal{O} to be *perfectoid* if we have the equivalence:

$$\mathcal{O}^{\text{preperfectoidization}} \times A/I \xrightarrow{\sim} \mathcal{O}. \quad (3.4.97)$$

¹⁷Before the Ben-Bassat-Mukherjee p -adic formal analytification we take the corresponding derived (p, I) -completion.

3.4.3 Functional Analytic Derived de Rham Complexes for $(\infty, 1)$ -Analytic Stacks and de Rham Preperfectoidizations

As in [LL] we have the comparison between the derived prismatic cohomology and the corresponding derived de Rham cohomology in some very well-defined way which respects the corresponding filtrations, we can then in our situation take the corresponding definition of some derived de Rham complex as the one side of the comparison from [LL]¹⁸. To be more precise after [An1, Chapitre 3], [An2], [B1, Chapter 2, Chapter 8], [Bei, Chapter 1], [G1, Chapter 5], [GL, Chapter 3, Chapter 4], [Ill1, Chapitre II, Chapitre III], [Ill2, Chapitre VIII], [Qui, Section 4], and [BMS2, Example 5.11, Example 5.12] we define the corresponding:

Definition 3.4.7. Then we can in the same fashion consider the corresponding derived de Rham complex presheaves for the commutative algebras as in the above (for a given prism (A, I)):

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{?/A}, \quad (3.4.98)$$

¹⁹by the regular corresponding left Kan extension techniques on the following $(\infty, 1)$ -compactly generated closures of the corresponding polynomials given over A/I with a chosen prism (A, I) :

$$\mathrm{Proj}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndSeminormed}_{A/I}) / \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndSeminormed}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology, homotopyepimorphisms}} \quad (3.4.99)$$

$$\mathrm{Proj}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m\mathrm{Seminormed}_{A/I}) / \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m\mathrm{Seminormed}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology, homotopyepimorphisms}} \quad (3.4.100)$$

$$\mathrm{Proj}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndNormed}_{A/I}) / \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndNormed}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology, homotopyepimorphisms}} \quad (3.4.101)$$

$$\mathrm{Proj}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m\mathrm{Normed}_{A/I}) / \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m\mathrm{Normed}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology, homotopyepimorphisms}} \quad (3.4.102)$$

$$\mathrm{Proj}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndBanach}_{A/I}) / \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndBanach}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology, homotopyepimorphisms}} \quad (3.4.103)$$

$$\mathrm{Proj}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m\mathrm{Banach}_{A/I}) / \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m\mathrm{Banach}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology, homotopyepimorphisms}} \quad (3.4.104)$$

We call the corresponding functors functional analytic derived de Rham complex presheaves which we are going to denote that as in the following:

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{?/A, \text{functionalanalytic, KKM, BBM, formalanalytification}} \cdot \quad (3.4.105)$$

¹⁸Our goal here is actually study the corresponding derived de Rham period rings and the corresponding applications in p -adic Hodge theory extending work of [DLLZ1], [DLLZ2], [Sch2] in the motivation from [GL], when applying the construction to derived p -adic formal stacks and derived logarithmic p -adic formal stacks.

¹⁹After taking derived p -completion.

This would mean the following definition²⁰:

$$\text{Kan}_{\text{Left}}\text{deRham}_{?/A, \text{functionalanalytic}, \text{KKM}, \text{BBM}, \text{formalanalytification}}(\mathcal{O}) \quad (3.4.106)$$

$$:= \left(\text{homotopycolimit}_{i \text{ sifted, derivedcategory}_{\infty}(A/I\text{-Module})} \text{Kan}_{\text{Left}}\text{deRham}_{?/A, \text{functionalanalytic}, \text{KKM}} \right) \quad (3.4.107)$$

$$(\mathcal{O}_i)^{\wedge} \quad (3.4.108)$$

$$\text{BBM, formalanalytification} \quad (3.4.109)$$

by writing any object \mathcal{O} as the corresponding colimit

$$\text{homotopycolimit}_{i \text{ sifted}} \mathcal{O}_i.$$

These are quite large $(\infty, 1)$ -commutative ring objects in the corresponding $(\infty, 1)$ -categories for $R = A/I$:

$$\text{Ind}\# \text{Quasicoherent}^{\text{presheaf}}_{\text{Ind}^{\text{smoothformalseriesclosure}}_{\infty}\text{-Toposes}} \text{ringed, commutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_R) \text{Commutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism} \quad (3.4.110)$$

$$\text{Ind}\# \text{Quasicoherent}^{\text{presheaf}}_{\text{Ind}^{\text{smoothformalseriesclosure}}_{\infty}\text{-Toposes}} \text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_R) \text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism} \quad (3.4.111)$$

$$\text{Ind}\# \text{Quasicoherent}^{\text{presheaf}}_{\text{Ind}^{\text{smoothformalseriesclosure}}_{\infty}\text{-Toposes}} \text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{IndNormed}_R) \text{Commutativealgebra}_{\text{simplicial}}(\text{IndNormed}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism} \quad (3.4.112)$$

$$\text{Ind}\# \text{Quasicoherent}^{\text{presheaf}}_{\text{Ind}^{\text{smoothformalseriesclosure}}_{\infty}\text{-Toposes}} \text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_R) \text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism} \quad (3.4.113)$$

$$\text{Ind}\# \text{Quasicoherent}^{\text{presheaf}}_{\text{Ind}^{\text{smoothformalseriesclosure}}_{\infty}\text{-Toposes}} \text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{IndBanach}_R) \text{Commutativealgebra}_{\text{simplicial}}(\text{IndBanach}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism} \quad (3.4.114)$$

$$\text{Ind}\# \text{Quasicoherent}^{\text{presheaf}}_{\text{Ind}^{\text{smoothformalseriesclosure}}_{\infty}\text{-Toposes}} \text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_R) \text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism} \quad (3.4.115)$$

$$(3.4.116)$$

after taking the formal series ring left Kan extension analytification from [BBM, Section 4.2], which is defined by taking the left Kan extension to all the $(\infty, 1)$ -ring objects in the ∞ -derived category of all A -modules from formal series rings over A .

²⁰Before the Ben-Bassat-Mukherjee p -adic formal analytification we take the corresponding derived (p, I) -completion.

Now we apply this construction to any A/I formal scheme (X, \mathcal{O}_X) . But in order to use prismatic technology to reach the corresponding period derived de Rham sheaves, we need to consider more, this would be some definition for ‘perfectoidizations’:

Definition 3.4.8. Let (A, I) be a perfectoid prism, and we consider any E_∞ -ring \mathcal{O} in the following

$$\text{Proj}^{\text{smoothformalseriesclosure}}_\infty - \text{Toposes}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_{A/I})^{\text{opposite}}, \text{Grothendiecktopology}, \text{homotopyepimorphisms}}^{\text{ringed, commutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_{A/I})} \quad (3.4.117)$$

$$\text{Proj}^{\text{smoothformalseriesclosure}}_\infty - \text{Toposes}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_{A/I})^{\text{opposite}}, \text{Grothendiecktopology}, \text{homotopyepimorphisms}}^{\text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_{A/I})} \quad (3.4.118)$$

$$\text{Proj}^{\text{smoothformalseriesclosure}}_\infty - \text{Toposes}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{IndNormed}_{A/I})^{\text{opposite}}, \text{Grothendiecktopology}, \text{homotopyepimorphisms}}^{\text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{IndNormed}_{A/I})} \quad (3.4.119)$$

$$\text{Proj}^{\text{smoothformalseriesclosure}}_\infty - \text{Toposes}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_{A/I})^{\text{opposite}}, \text{Grothendiecktopology}, \text{homotopyepimorphisms}}^{\text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_{A/I})} \quad (3.4.120)$$

$$\text{Proj}^{\text{smoothformalseriesclosure}}_\infty - \text{Toposes}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{IndBanach}_{A/I})^{\text{opposite}}, \text{Grothendiecktopology}, \text{homotopyepimorphisms}}^{\text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{IndBanach}_{A/I})} \quad (3.4.121)$$

$$\text{Proj}^{\text{smoothformalseriesclosure}}_\infty - \text{Toposes}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_{A/I})^{\text{opposite}}, \text{Grothendiecktopology}, \text{homotopyepimorphisms}}^{\text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_{A/I})} \quad (3.4.122)$$

Then consider the derived prismatic object:

$$\text{Kan}_{\text{Left}} \text{deRham}_{?/A, \text{functionalanalytic}, \text{KKM}, \text{BBM}, \text{formalanalytification}}(\mathcal{O}). \quad (3.4.123)$$

Then as in [BS1, Definition 8.2] we have the following de Rham preperfectoidization:

$$(\mathcal{O})^{\text{deRham, preperfectoidization}} \quad (3.4.124)$$

$$:= \text{Colimit}(\text{Kan}_{\text{Left}} \text{deRham}_{?/A, \text{functionalanalytic}, \text{KKM}, \text{BBM}, \text{formalanalytification}}(\mathcal{O}) \rightarrow \quad (3.4.125)$$

$$\text{Fro}_* \text{Kan}_{\text{Left}} \text{deRham}_{?/A, \text{functionalanalytic}, \text{KKM}, \text{BBM}, \text{formalanalytification}}(\mathcal{O}) \quad (3.4.126)$$

$$\rightarrow \text{Fro}_* \text{Fro}_* \text{Kan}_{\text{Left}} \text{deRham}_{?/A, \text{functionalanalytic}, \text{KKM}, \text{BBM}, \text{formalanalytification}}(\mathcal{O}) \rightarrow \dots \quad (3.4.127)$$

$$\text{BBM, formalanalytification}, \quad (3.4.128)$$

after taking the formal series ring left Kan extension analytification from [BBM, Section 4.2], which is defined by taking the left Kan extension to all the $(\infty, 1)$ -ring object in the ∞ -derived category of all A/I -modules from formal series rings over A/I . Furthermore one can take derived (p, I) -completion to achieve the derived (p, I) -completed versions.

3.4.4 Functional Analytic Derived Prismatic Complexes for Inductive $(\infty, 1)$ -Analytic Stacks

We now promote the construction in the previous sections to the corresponding inductive systems of $(\infty, 1)$ -ringed toposes level after Lurie [Lu1], [Lu2] and [Lu3] in the ∞ -category

of ∞ -ringed toposes, Bambozzi-Ben-Bassat-Kremnizer [BBBK], Ben-Bassat-Mukherjee [BBM], Bambozzi-Kremnizer [BK], Clausen-Scholze [CS1] [CS2] and Kelly-Kremnizer-Mukherjee [KKM] in the ∞ -category of ∞ -functional analytic ringed toposes.

Now we consider the following ∞ -categories of the corresponding ∞ -analytic ringed toposes from Bambozzi-Ben-Bassat-Kremnizer [BBBK]:

$$\infty - \text{Toposes}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_?)^{\text{opposite}}, \text{Grothendiecktopology, homotopyepimorphism}'}^{\text{ringed, commutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_?)}$$
 (3.4.129)

$$\infty - \text{Toposes}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_?)^{\text{opposite}}, \text{Grothendiecktopology, homotopyepimorphism}'}^{\text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_?)}$$
 (3.4.130)

$$\infty - \text{Toposes}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{IndNormed}_?)^{\text{opposite}}, \text{Grothendiecktopology, homotopyepimorphism}'}^{\text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{IndNormed}_?)}$$
 (3.4.131)

$$\infty - \text{Toposes}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_?)^{\text{opposite}}, \text{Grothendiecktopology, homotopyepimorphism}'}^{\text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_?)}$$
 (3.4.132)

$$\infty - \text{Toposes}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{IndBanach}_?)^{\text{opposite}}, \text{Grothendiecktopology, homotopyepimorphism}'}^{\text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{IndBanach}_?)}$$
 (3.4.133)

$$\infty - \text{Toposes}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_?)^{\text{opposite}}, \text{Grothendiecktopology, homotopyepimorphism}'}^{\text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_?)}$$
 (3.4.134)

$$\text{Ind}^{\text{smoothformalseriesclosure}} \infty - \text{Toposes}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_?)^{\text{opposite}}, \text{Grothendiecktopology, homotopyepimorphism}'}^{\text{ringed, commutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_?)}$$
 (3.4.135)

$$\text{Ind}^{\text{smoothformalseriesclosure}} \infty - \text{Toposes}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_?)^{\text{opposite}}, \text{Grothendiecktopology, homotopyepimorphism}'}^{\text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_?)}$$
 (3.4.136)

$$\text{Ind}^{\text{smoothformalseriesclosure}} \infty - \text{Toposes}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{IndNormed}_?)^{\text{opposite}}, \text{Grothendiecktopology, homotopyepimorphism}'}^{\text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{IndNormed}_?)}$$
 (3.4.137)

$$\text{Ind}^{\text{smoothformalseriesclosure}} \infty - \text{Toposes}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_?)^{\text{opposite}}, \text{Grothendiecktopology, homotopyepimorphism}'}^{\text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_?)}$$
 (3.4.138)

$$\text{Ind}^{\text{smoothformalseriesclosure}} \infty - \text{Toposes}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{IndBanach}_?)^{\text{opposite}}, \text{Grothendiecktopology, homotopyepimorphism}'}^{\text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{IndBanach}_?)}$$
 (3.4.139)

$$\text{Ind}^{\text{smoothformalseriesclosure}} \infty - \text{Toposes}_{\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_?)^{\text{opposite}}, \text{Grothendiecktopology, homotopyepimorphism}'}^{\text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_?)}$$
 (3.4.140)

$$(3.4.141)$$

Definition 3.4.9. One can actually define the derived prismatic cohomology presheaves through derived topological Hochschild cohomology presheaves, derived topological period cohomology presheaves and derived topological cyclic cohomology presheaves as in [BMS, Section 2.2, Section 2.3], [BS1, Theorem 1.13]:

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{THH}, \mathrm{Kan}_{\mathrm{Left}}\mathrm{TP}, \mathrm{Kan}_{\mathrm{Left}}\mathrm{TC}, \quad (3.4.142)$$

on the following $(\infty, 1)$ -compactly generated closures of the corresponding polynomials²¹ given over A/I with a chosen prism (A, I) ²²:

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndSeminormed}_{A/I})}^{\mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndSeminormed}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology}, \mathrm{homotopyepimorph}} \quad (3.4.143)$$

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m\mathrm{Seminormed}_{A/I})}^{\mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m\mathrm{Seminormed}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology}, \mathrm{homotopyepimorph}} \quad (3.4.144)$$

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndNormed}_{A/I})}^{\mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndNormed}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology}, \mathrm{homotopyepimorph}} \quad (3.4.145)$$

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m\mathrm{Normed}_{A/I})}^{\mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m\mathrm{Normed}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology}, \mathrm{homotopyepimorph}} \quad (3.4.146)$$

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndBanach}_{A/I})}^{\mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndBanach}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology}, \mathrm{homotopyepimorph}} \quad (3.4.147)$$

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m\mathrm{Banach}_{A/I})}^{\mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m\mathrm{Banach}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology}, \mathrm{homotopyepimorph}} \quad (3.4.148)$$

We call the corresponding functors are derived functional analytic Hochschild cohomology presheaves, derived functional analytic period cohomology presheaves and derived functional analytic cyclic cohomology presheaves, which we are going to denote these presheaves as in

²¹Definitely, we need to put certain norms over in some relatively canonical way, as in [BBM, Section 4.2] one can basically consider rigid ones and dagger ones, and so on. We restrict to the *formal* one.

²²In all the following, we assume this prism to be bounded and satisfy that A/I is Banach.

the following for any ∞ -ringed topos $(\mathbb{X}, \mathcal{O}) = \mathop{\mathrm{homotopycolimit}}_i(\mathbb{X}_i, \mathcal{O}_i)$:

$$\mathrm{Kan}_{\mathrm{Left}} \mathrm{THH}_{\mathrm{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \mathrm{formalanalytification}}(\mathcal{O}) := \quad (3.4.149)$$

$$\left(\mathop{\mathrm{homotopylimit}}_i \quad \mathrm{Kan}_{\mathrm{Left}} \mathrm{THH}_{\mathrm{functionalanalytic}, \mathrm{KKM}}(\mathcal{O}_i) \right)_{\mathrm{sifted}, \mathrm{derivedcategory}_{\infty}(A/I\text{-Module})} \quad (3.4.150)$$

$$\left. \right)_{\mathrm{BBM}, \mathrm{formalanalytification}} \quad (3.4.151)$$

$$\mathrm{Kan}_{\mathrm{Left}} \mathrm{TP}_{\mathrm{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \mathrm{formalanalytification}}(\mathcal{O}) := \quad (3.4.152)$$

$$\left(\mathop{\mathrm{homotopylimit}}_i \quad \mathrm{Kan}_{\mathrm{Left}} \mathrm{TP}_{\mathrm{functionalanalytic}, \mathrm{KKM}}(\mathcal{O}_i) \right)_{\mathrm{sifted}, \mathrm{derivedcategory}_{\infty}(A/I\text{-Module})} \quad (3.4.153)$$

$$\left. \right)_{\mathrm{BBM}, \mathrm{formalanalytification}} \quad (3.4.154)$$

$$\mathrm{Kan}_{\mathrm{Left}} \mathrm{TC}_{\mathrm{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \mathrm{formalanalytification}}(\mathcal{O}) := \quad (3.4.155)$$

$$\left(\mathop{\mathrm{homotopylimit}}_i \quad \mathrm{Kan}_{\mathrm{Left}} \mathrm{TC}_{\mathrm{functionalanalytic}, \mathrm{KKM}}(\mathcal{O}_i) \right)_{\mathrm{sifted}, \mathrm{derivedcategory}_{\infty}(A/I\text{-Module})} \quad (3.4.156)$$

$$\left. \right)_{\mathrm{BBM}, \mathrm{formalanalytification}} \quad (3.4.157)$$

by writing any object \mathcal{O} as the corresponding limit²³

$$\mathop{\mathrm{homotopylimit}}_i \quad \mathcal{O}_i := \mathop{\mathrm{homotopylimit}}_i \quad \mathcal{O}_{\mathrm{Spectrumadic}_{\mathrm{BK}}(\mathrm{Formalseriesring}_i)}$$

These are quite large $(\infty, 1)$ -commutative ring objects in the corresponding $(\infty, 1)$ -categories for $? = A/I$ over $(\mathbb{X}, \mathcal{O})$:

$$\mathrm{Ind}\#_{\mathrm{Quasicoherent}}^{\mathrm{presheaf}} \quad \mathrm{Ind}_{\mathrm{smoothformalseriesclosure}_{\infty}\text{-Toposes}}^{\mathrm{ringed}, \mathrm{commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndSeminormed}_?)^{\mathrm{opposite}}, \mathrm{Grotopology}, \mathrm{homotopyepimorphism}} \quad (3.4.158)$$

$$\mathrm{Ind}\#_{\mathrm{Quasicoherent}}^{\mathrm{presheaf}} \quad \mathrm{Ind}_{\mathrm{smoothformalseriesclosure}_{\infty}\text{-Toposes}}^{\mathrm{ringed}, \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Seminormed}_?)^{\mathrm{opposite}}, \mathrm{Grotopology}, \mathrm{homotopyepimorphism}} \quad (3.4.159)$$

$$\mathrm{Ind}\#_{\mathrm{Quasicoherent}}^{\mathrm{presheaf}} \quad \mathrm{Ind}_{\mathrm{smoothformalseriesclosure}_{\infty}\text{-Toposes}}^{\mathrm{ringed}, \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndNormed}_?)^{\mathrm{opposite}}, \mathrm{Grotopology}, \mathrm{homotopyepimorphism}} \quad (3.4.160)$$

$$\mathrm{Ind}\#_{\mathrm{Quasicoherent}}^{\mathrm{presheaf}} \quad \mathrm{Ind}_{\mathrm{smoothformalseriesclosure}_{\infty}\text{-Toposes}}^{\mathrm{ringed}, \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Normed}_?)^{\mathrm{opposite}}, \mathrm{Grotopology}, \mathrm{homotopyepimorphism}} \quad (3.4.161)$$

$$\mathrm{Ind}\#_{\mathrm{Quasicoherent}}^{\mathrm{presheaf}} \quad \mathrm{Ind}_{\mathrm{smoothformalseriesclosure}_{\infty}\text{-Toposes}}^{\mathrm{ringed}, \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndBanach}_?)^{\mathrm{opposite}}, \mathrm{Grotopology}, \mathrm{homotopyepimorphism}} \quad (3.4.162)$$

$$\mathrm{Ind}\#_{\mathrm{Quasicoherent}}^{\mathrm{presheaf}} \quad \mathrm{Ind}_{\mathrm{smoothformalseriesclosure}_{\infty}\text{-Toposes}}^{\mathrm{ringed}, \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Banach}_?)^{\mathrm{opposite}}, \mathrm{Grotopology}, \mathrm{homotopyepimorphism}} \quad (3.4.163)$$

$$(3.4.164)$$

²³Note that we are working with ind- ∞ -ringed ∞ -stacks.

after taking the formal series ring left Kan extension analytification from [BBM, Section 4.2], which is defined by taking the left Kan extension to all the $(\infty, 1)$ -ring objects in the ∞ -derived category of all A -modules from formal series rings over A .

Definition 3.4.10. Then we can in the same fashion consider the corresponding derived prismatic complex presheaves [BS1, Construction 7.6]²⁴ for the commutative algebras as in the above (for a given prism (A, I)):

$$\mathrm{Kan}_{\mathrm{Left}}\Delta_{?/A}, \quad (3.4.165)$$

by the regular corresponding left Kan extension techniques on the following $(\infty, 1)$ -compactly generated closures of the corresponding polynomials given over A/I with a chosen prism (A, I) :

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndSeminormed}_{A/I})} \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndSeminormed}_{A/I})^{\mathrm{opposite}, \mathrm{Grothendiecktopology}, \mathrm{homotopyepimorph}} \quad (3.4.166)$$

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Seminormed}_{A/I})} \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Seminormed}_{A/I})^{\mathrm{opposite}, \mathrm{Grothendiecktopology}, \mathrm{homotopyepimorph}} \quad (3.4.167)$$

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndNormed}_{A/I})} \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndNormed}_{A/I})^{\mathrm{opposite}, \mathrm{Grothendiecktopology}, \mathrm{homotopyepimorph}} \quad (3.4.168)$$

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Normed}_{A/I})} \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Normed}_{A/I})^{\mathrm{opposite}, \mathrm{Grothendiecktopology}, \mathrm{homotopyepimorph}} \quad (3.4.169)$$

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndBanach}_{A/I})} \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndBanach}_{A/I})^{\mathrm{opposite}, \mathrm{Grothendiecktopology}, \mathrm{homotopyepimorph}} \quad (3.4.170)$$

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Banach}_{A/I})} \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Banach}_{A/I})^{\mathrm{opposite}, \mathrm{Grothendiecktopology}, \mathrm{homotopyepimorph}} \quad (3.4.171)$$

We call the corresponding functors functional analytic derived prismatic complex presheaves which we are going to denote that as in the following:

$$\mathrm{Kan}_{\mathrm{Left}}\Delta_{?/A, \mathrm{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \mathrm{formalanalytification}} \quad (3.4.172)$$

²⁴One just applies [BS1, Construction 7.6] and then takes the left Kan extensions.

This would mean the following definition²⁵:

$$\text{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, KKM, BBM, formalanalytification}}(\mathcal{O}) \quad (3.4.173)$$

$$:= \left(\underset{i}{\text{homotopylimit}} \underset{\text{sifted, derivedcategory}_{\infty}(A/I\text{-Module})}{\text{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, KKM}}(\mathcal{O}_i)} \right)^{\wedge} \quad (3.4.174)$$

$$\text{BBM, formalanalytification} \quad (3.4.175)$$

by writing any object \mathcal{O} as the corresponding limit²⁶

$$\underset{i}{\text{homotopylimit}} \underset{\text{sifted}}{\mathcal{O}_i}.$$

These are quite large $(\infty, 1)$ -commutative ring objects in the corresponding $(\infty, 1)$ -categories for $R = A/I$:

$$\text{Ind}\# \text{Quasicoherent}^{\text{presheaf}} \underset{\text{Ind}^{\text{smoothformalseriesclosure}}}{\infty\text{-Toposes}} \underset{\text{ringed, commutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_R)}{\text{Commutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}} \cdot \quad (3.4.179)$$

$$\text{Ind}\# \text{Quasicoherent}^{\text{presheaf}} \underset{\text{Ind}^{\text{smoothformalseriesclosure}}}{\infty\text{-Toposes}} \underset{\text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_R)}{\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}} \cdot \quad (3.4.180)$$

$$\text{Ind}\# \text{Quasicoherent}^{\text{presheaf}} \underset{\text{Ind}^{\text{smoothformalseriesclosure}}}{\infty\text{-Toposes}} \underset{\text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{IndNormed}_R)}{\text{Commutativealgebra}_{\text{simplicial}}(\text{IndNormed}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}} \cdot \quad (3.4.181)$$

$$\text{Ind}\# \text{Quasicoherent}^{\text{presheaf}} \underset{\text{Ind}^{\text{smoothformalseriesclosure}}}{\infty\text{-Toposes}} \underset{\text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_R)}{\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}} \cdot \quad (3.4.182)$$

$$\text{Ind}\# \text{Quasicoherent}^{\text{presheaf}} \underset{\text{Ind}^{\text{smoothformalseriesclosure}}}{\infty\text{-Toposes}} \underset{\text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{IndBanach}_R)}{\text{Commutativealgebra}_{\text{simplicial}}(\text{IndBanach}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}} \cdot \quad (3.4.183)$$

$$\text{Ind}\# \text{Quasicoherent}^{\text{presheaf}} \underset{\text{Ind}^{\text{smoothformalseriesclosure}}}{\infty\text{-Toposes}} \underset{\text{ringed, Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_R)}{\text{Commutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}} \cdot \quad (3.4.184)$$

$$(3.4.185)$$

²⁵Before the Ben-Bassat-Mukherjee p -adic formal analytification we take the corresponding derived (p, I) -completion.

²⁶Here we remind the readers of the corresponding foundation here, namely the presheaf \mathcal{O}_i in fact takes the value in Koszul complex taking the following form:

$$\text{Koszulcomplex}_{A/I\langle X_1, \dots, X_l \rangle \langle T_1, \dots, T_m \rangle}(a_1 - T_1 b_1, \dots, a_m - T_m b_m) \quad (3.4.176)$$

$$= A/I\langle X_1, \dots, X_l \rangle \langle T_1, \dots, T_m \rangle / \mathbb{L}(a_1 - T_1 b_1, \dots, a_m - T_m b_m). \quad (3.4.177)$$

This is actually derived p -complete since each homotopy group is derived p -complete (from the corresponding Banach structure from $A/I\langle X_1, \dots, X_l \rangle$ induced from the p -adic topology). Namely the definition of the presheaf:

$$\text{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, KKM}}(\mathcal{O}_i) \quad (3.4.178)$$

is directly the application of the derived prismatic functor from [BS1, Construction 7.6].

after taking the formal series ring left Kan extension analytification from [BBM, Section 4.2], which is defined by taking the left Kan extension to all the $(\infty, 1)$ -ring objects in the ∞ -derived category of all A -modules from formal series rings over A .

3.4.5 Functional Analytic Derived Preperfectoidizations for Inductive $(\infty, 1)$ -Analytic Stacks

Then as in [BS1, Definition 8.2] we consider the corresponding perfectoidization in this analytic setting.

Definition 3.4.11. Let (A, I) be a perfectoid prism, and we consider any E_∞ -ring \mathcal{O} in the following

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}} \infty - \mathrm{Toposes}^{\mathrm{ringed, commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndSeminormed}_{A/I}) / \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndSeminormed}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology, homotopyepimorph}} \quad (3.4.186)$$

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}} \infty - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Seminormed}_{A/I}) / \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Seminormed}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology, homotopyepimorph}} \quad (3.4.187)$$

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}} \infty - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndNormed}_{A/I}) / \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndNormed}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology, homotopyepimorph}} \quad (3.4.188)$$

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}} \infty - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Normed}_{A/I}) / \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Normed}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology, homotopyepimorph}} \quad (3.4.189)$$

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}} \infty - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndBanach}_{A/I}) / \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndBanach}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology, homotopyepimorph}} \quad (3.4.190)$$

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}} \infty - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Banach}_{A/I}) / \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Banach}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology, homotopyepimorph}} \quad (3.4.191)$$

Then consider the derived prismatic object:

$$\mathrm{Kan}_{\mathrm{Left} \Delta_{?/A, \text{functionalanalytic, KKM, BBM, formalanalytification}}(\mathcal{O})}. \quad (3.4.192)$$

Then as in [BS1, Definition 8.2] we have the following preperfectoidization:

$$(\mathcal{O})^{\mathrm{preperfectoidization}} \quad (3.4.193)$$

$$:= \mathrm{Colimit}(\mathrm{Kan}_{\mathrm{Left} \Delta_{?/A, \text{functionalanalytic, KKM, BBM, formalanalytification}}(\mathcal{O}) \rightarrow \quad (3.4.194)$$

$$\mathrm{Fro}_* \mathrm{Kan}_{\mathrm{Left} \Delta_{?/A, \text{functionalanalytic, KKM, BBM, formalanalytification}}(\mathcal{O}) \quad (3.4.195)$$

$$\rightarrow \mathrm{Fro}_* \mathrm{Fro}_* \mathrm{Kan}_{\mathrm{Left} \Delta_{?/A, \text{functionalanalytic, KKM, BBM, formalanalytification}}(\mathcal{O}) \rightarrow \dots)^{\mathrm{BBM, formalanalytification}}, \quad (3.4.196)$$

after taking the formal series ring left Kan extension analytification from [BBM, Section 4.2], which is defined by taking the left Kan extension to all the $(\infty, 1)$ -ring object in the ∞ -derived category of all A -modules Δ from formal series rings over A . Then we define the corresponding

perfectoidization:

$$(\mathcal{O})^{\text{perfectoidization}} \tag{3.4.197}$$

$$:= \text{Colimit}(\text{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, KKM, BBM, formalanalytification}}(\mathcal{O}) \longrightarrow \tag{3.4.198}$$

$$\text{Fro}_*\text{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, KKM, BBM, formalanalytification}}(\mathcal{O}) \tag{3.4.199}$$

$$\longrightarrow \text{Fro}_*\text{Fro}_*\text{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, KKM, BBM, formalanalytification}}(\mathcal{O}) \longrightarrow \dots)^{\text{BBM, formalanalytification}} \times A/I. \tag{3.4.200}$$

Furthermore one can take derived (p, I) -completion to achieve the derived (p, I) -completed versions:

$$\mathcal{O}^{\text{preperfectoidization, derivedcomplete}} := (\mathcal{O}^{\text{preperfectoidization}})^\wedge, \tag{3.4.201}$$

$$\mathcal{O}^{\text{perfectoidization, derivedcomplete}} := \mathcal{O}^{\text{preperfectoidization, derivedcomplete}} \times A/I. \tag{3.4.202}$$

$$\tag{3.4.203}$$

These are large $(\infty, 1)$ -commutative algebra objects in the corresponding categories as in the above, attached to also large $(\infty, 1)$ -commutative algebra objects. When we apply this to the corresponding sub- $(\infty, 1)$ -categories of Banach perfectoid objects in [BMS2], [GR], [KL1], [KL2], [Ked1], [Sch3], we will recover the corresponding distinguished elemental deformation processes defined in [BMS2], [GR], [KL1], [KL2], [Ked1], [Sch3].

Remark 3.4.12. One can then define such ring \mathcal{O} to be *preperfectoid* if we have the equivalence:

$$\mathcal{O}^{\text{preperfectoidization}} \xrightarrow{\sim} \mathcal{O}. \tag{3.4.204}$$

One can then define such ring \mathcal{O} to be *perfectoid* if we have the equivalence:

$$\mathcal{O}^{\text{preperfectoidization}} \times A/I \xrightarrow{\sim} \mathcal{O}. \tag{3.4.205}$$

3.4.6 Functional Analytic Derived de Rham Complexes for Inductive $(\infty, 1)$ -Analytic Stacks and de Rham Preperfectoidizations

As in [LL] we have the comparison between the derived prismatic cohomology and the corresponding derived de Rham cohomology in some very well-defined way which respects the corresponding filtrations, we can then in our situation take the corresponding definition of some derived de Rham complex as the one side of the comparison from [LL]²⁷. To be more precise after [An1, Chapitre 3], [An2], [B1, Chapter 2, Chapter 8], [Bei, Chapter 1], [G1, Chapter 5], [GL, Chapter 3, Chapter 4], [Ill1, Chapitre II, Chapitre III], [Ill2, Chapitre VIII], [Qui, Section 4], and [BMS2, Example 5.11, Example 5.12] we define the corresponding:

Definition 3.4.13. Then we can in the same fashion consider the corresponding derived de Rham complex presheaves for the commutative algebras as in the above (for a given prism (A, I)):

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{?/A}, \quad (3.4.206)$$

²⁸by the regular corresponding left Kan extension techniques on the following $(\infty, 1)$ -compactly generated closures of the corresponding polynomials given over A/I with a chosen prism (A, I) :

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndSeminormed}_{A/I})} \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndSeminormed}_{A/I})^{\mathrm{opposite}, \mathrm{Grothendiecktopology}, \mathrm{homotopyepimorph}} \quad (3.4.207)$$

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Seminormed}_{A/I})} \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Seminormed}_{A/I})^{\mathrm{opposite}, \mathrm{Grothendiecktopology}, \mathrm{homotopyepimorph}} \quad (3.4.208)$$

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndNormed}_{A/I})} \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndNormed}_{A/I})^{\mathrm{opposite}, \mathrm{Grothendiecktopology}, \mathrm{homotopyepimorph}} \quad (3.4.209)$$

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Normed}_{A/I})} \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Normed}_{A/I})^{\mathrm{opposite}, \mathrm{Grothendiecktopology}, \mathrm{homotopyepimorph}} \quad (3.4.210)$$

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndBanach}_{A/I})} \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndBanach}_{A/I})^{\mathrm{opposite}, \mathrm{Grothendiecktopology}, \mathrm{homotopyepimorph}} \quad (3.4.211)$$

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}}_{\infty} - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Banach}_{A/I})} \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Banach}_{A/I})^{\mathrm{opposite}, \mathrm{Grothendiecktopology}, \mathrm{homotopyepimorph}} \quad (3.4.212)$$

We call the corresponding functors functional analytic derived de Rham complex presheaves which we are going to denote that as in the following:

$$\mathrm{Kan}_{\mathrm{Left}}\mathrm{deRham}_{?/A, \mathrm{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \mathrm{formalanalytification}}. \quad (3.4.213)$$

²⁷Our goal here is actually study the corresponding derived de Rham period rings and the corresponding applications in p -adic Hodge theory extending work of [DLLZ1], [DLLZ2], [Sch2] in the motivation from [GL], when applying the construction to derived p -adic formal stacks and derived logarithmic p -adic formal stacks.

²⁸After taking derived p -completion.

This would mean the following definition²⁹:

$$\text{Kan}_{\text{Left}}\text{deRham}_{?/A, \text{functionalanalytic}, \text{KKM}, \text{BBM}, \text{formalanalytification}}(\mathcal{O}) \quad (3.4.214)$$

$$:= \left(\underset{i}{\text{homotopylimit}} \underset{\text{sifted, derived category}_{\infty}(A/I\text{-Module})}{\text{Kan}_{\text{Left}}\text{deRham}_{?/A, \text{functionalanalytic}, \text{KKM}}(\mathcal{O}_i)} \right) \quad (3.4.215)$$

$$(\mathcal{O}_i)^{\wedge} \quad (3.4.216)$$

$$\text{BBM, formalanalytification} \quad (3.4.217)$$

by writing any object \mathcal{O} as the corresponding colimit

$$\underset{i}{\text{homotopylimit}} \mathcal{O}_i.$$

These are quite large $(\infty, 1)$ -commutative ring objects in the corresponding $(\infty, 1)$ -categories for $R = A/I$:

$$\text{Ind}\#_{\text{Quasicoherent}}^{\text{presheaf}} \underset{\text{Ind}^{\text{smoothformalseriesclosure}}}{\infty\text{-Toposes}} \underset{\text{ringed, commutative algebra}_{\text{simplicial}}(\text{IndSeminormed}_R)}{\text{Commutative algebra}_{\text{simplicial}}(\text{IndSeminormed}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}} \quad (3.4.218)$$

$$\text{Ind}\#_{\text{Quasicoherent}}^{\text{presheaf}} \underset{\text{Ind}^{\text{smoothformalseriesclosure}}}{\infty\text{-Toposes}} \underset{\text{ringed, Commutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_R)}{\text{Commutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}} \quad (3.4.219)$$

$$\text{Ind}\#_{\text{Quasicoherent}}^{\text{presheaf}} \underset{\text{Ind}^{\text{smoothformalseriesclosure}}}{\infty\text{-Toposes}} \underset{\text{ringed, Commutative algebra}_{\text{simplicial}}(\text{IndNormed}_R)}{\text{Commutative algebra}_{\text{simplicial}}(\text{IndNormed}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}} \quad (3.4.220)$$

$$\text{Ind}\#_{\text{Quasicoherent}}^{\text{presheaf}} \underset{\text{Ind}^{\text{smoothformalseriesclosure}}}{\infty\text{-Toposes}} \underset{\text{ringed, Commutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_R)}{\text{Commutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}} \quad (3.4.221)$$

$$\text{Ind}\#_{\text{Quasicoherent}}^{\text{presheaf}} \underset{\text{Ind}^{\text{smoothformalseriesclosure}}}{\infty\text{-Toposes}} \underset{\text{ringed, Commutative algebra}_{\text{simplicial}}(\text{IndBanach}_R)}{\text{Commutative algebra}_{\text{simplicial}}(\text{IndBanach}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}} \quad (3.4.222)$$

$$\text{Ind}\#_{\text{Quasicoherent}}^{\text{presheaf}} \underset{\text{Ind}^{\text{smoothformalseriesclosure}}}{\infty\text{-Toposes}} \underset{\text{ringed, Commutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_R)}{\text{Commutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}} \quad (3.4.223)$$

$$(3.4.224)$$

after taking the formal series ring left Kan extension analytification from [BBM, Section 4.2], which is defined by taking the left Kan extension to all the $(\infty, 1)$ -ring objects in the ∞ -derived category of all A -modules from formal series rings over A .

²⁹Before the Ben-Bassat-Mukherjee p -adic formal analytification we take the corresponding derived (p, I) -completion.

Now we apply this construction to any A/I formal scheme (X, \mathcal{O}_X) . But in order to use prismatic technology to reach the corresponding period derived de Rham sheaves, we need to consider more, this would be some definition for ‘perfectoidizations’:

Definition 3.4.14. Let (A, I) be a perfectoid prism, and we consider any E_∞ -ring \mathcal{O} in the following

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}}_\infty - \mathrm{Toposes}^{\mathrm{ringed, commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndSeminormed}_{A/I})} \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndSeminormed}_{A/I})^{\mathrm{opposite}, \mathrm{Grothendiecktopology, homotopyepimorph}} \quad (3.4.225)$$

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}}_\infty - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Seminormed}_{A/I})} \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Seminormed}_{A/I})^{\mathrm{opposite}, \mathrm{Grothendiecktopology, homotopyepimorph}} \quad (3.4.226)$$

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}}_\infty - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndNormed}_{A/I})} \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndNormed}_{A/I})^{\mathrm{opposite}, \mathrm{Grothendiecktopology, homotopyepimorph}} \quad (3.4.227)$$

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}}_\infty - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Normed}_{A/I})} \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Normed}_{A/I})^{\mathrm{opposite}, \mathrm{Grothendiecktopology, homotopyepimorph}} \quad (3.4.228)$$

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}}_\infty - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndBanach}_{A/I})} \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndBanach}_{A/I})^{\mathrm{opposite}, \mathrm{Grothendiecktopology, homotopyepimorph}} \quad (3.4.229)$$

$$\mathrm{Ind}^{\mathrm{smoothformalseriesclosure}}_\infty - \mathrm{Toposes}^{\mathrm{ringed, Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Banach}_{A/I})} \mathrm{Commutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Banach}_{A/I})^{\mathrm{opposite}, \mathrm{Grothendiecktopology, homotopyepimorph}} \quad (3.4.230)$$

Then consider the derived prismatic object:

$$\mathrm{Kan}_{\mathrm{Left}} \mathrm{deRham}_{?/A, \mathrm{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \mathrm{formalanalytification}}(\mathcal{O}). \quad (3.4.231)$$

Then as in [BS1, Definition 8.2] we have the following de Rham preperfectoidization:

$$(\mathcal{O})^{\mathrm{deRham, preperfectoidization}} \quad (3.4.232)$$

$$:= \mathrm{Colimit}(\mathrm{Kan}_{\mathrm{Left}} \mathrm{deRham}_{?/A, \mathrm{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \mathrm{formalanalytification}}(\mathcal{O}) \rightarrow \quad (3.4.233)$$

$$\mathrm{Fro}_* \mathrm{Kan}_{\mathrm{Left}} \mathrm{deRham}_{?/A, \mathrm{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \mathrm{formalanalytification}}(\mathcal{O}) \quad (3.4.234)$$

$$\rightarrow \mathrm{Fro}_* \mathrm{Fro}_* \mathrm{Kan}_{\mathrm{Left}} \mathrm{deRham}_{?/A, \mathrm{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \mathrm{formalanalytification}}(\mathcal{O}) \rightarrow \dots \quad (3.4.235)$$

$$\mathrm{BBM}, \mathrm{formalanalytification}, \quad (3.4.236)$$

after taking the formal series ring left Kan extension analytification from [BBM, Section 4.2], which is defined by taking the left Kan extension to all the $(\infty, 1)$ -ring object in the ∞ -derived category of all A/I -modules from formal series rings over A/I . Furthermore one can take derived (p, I) -completion to achieve the derived (p, I) -completed versions.

3.5 Functional Analytic Noncommutative Motives, Noncommutative Primate Cohomologies and the Preperfectoidizations

We now work in the noncommutative setting after [Kon1], [Ta], [KR1] and [KR2], with some philosophy rooted in some noncommutative motives and the corresponding nonabelian applications in noncommutative analytic geometry in the derived sense, and the noncommutative analogs of the corresponding Riemann hypothesis and the corresponding Tamagawa number conjectures. The issue is certainly that the usual Frobenius map looks strange, which tells us of the fact that actually we need to consider really large objects such as the corresponding Topological Hochschild Homologies and the corresponding nearby objects. Here we choose to consider [NS] in order to apply the constructions to certain ∞ -rings, which we will call them Fukaya-Kato analytifications from [FK].

Now we consider the following ∞ -categories of the corresponding ∞ -analytic ringed toposes from Bambozzi-Ben-Bassat-Kremnizer [BBBK] in some parallel way:

$$\infty - \text{Toposes}_{\text{ringed, Noncommutative algebra}_{\text{simplicial}}(\text{IndSeminormed}_?) / \text{Noncommutative algebra}_{\text{simplicial}}(\text{IndSeminormed}_?)^{\text{opposite}}, \text{Grothendieck topology, homotopy epimorphism} \quad (3.5.1)$$

$$\infty - \text{Toposes}_{\text{ringed, Noncommutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_?) / \text{Noncommutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_?)^{\text{opposite}}, \text{Grothendieck topology, homotopy epimorphism} \quad (3.5.2)$$

$$\infty - \text{Toposes}_{\text{ringed, Noncommutative algebra}_{\text{simplicial}}(\text{IndNormed}_?) / \text{Noncommutative algebra}_{\text{simplicial}}(\text{IndNormed}_?)^{\text{opposite}}, \text{Grothendieck topology, homotopy epimorphism} \quad (3.5.3)$$

$$\infty - \text{Toposes}_{\text{ringed, Noncommutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_?) / \text{Noncommutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_?)^{\text{opposite}}, \text{Grothendieck topology, homotopy epimorphism} \quad (3.5.4)$$

$$\infty - \text{Toposes}_{\text{ringed, Noncommutative algebra}_{\text{simplicial}}(\text{IndBanach}_?) / \text{Noncommutative algebra}_{\text{simplicial}}(\text{IndBanach}_?)^{\text{opposite}}, \text{Grothendieck topology, homotopy epimorphism} \quad (3.5.5)$$

$$\infty - \text{Toposes}_{\text{ringed, Noncommutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_?) / \text{Noncommutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_?)^{\text{opposite}}, \text{Grothendieck topology, homotopy epimorphism} \quad (3.5.6)$$

$$\text{Proj}^{\text{smooth formal series closure}} \infty - \text{Toposes}_{\text{ringed, Noncommutative algebra}_{\text{simplicial}}(\text{IndSeminormed}_?) / \text{Noncommutative algebra}_{\text{simplicial}}(\text{IndSeminormed}_?)^{\text{opposite}}, \text{Grothendieck topology, homotopy epimorphism} \quad (3.5.7)$$

$$\text{Proj}^{\text{smooth formal series closure}} \infty - \text{Toposes}_{\text{ringed, Noncommutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_?) / \text{Noncommutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_?)^{\text{opposite}}, \text{Grothendieck topology, homotopy epimorphism} \quad (3.5.8)$$

$$\text{Proj}^{\text{smooth formal series closure}} \infty - \text{Toposes}_{\text{ringed, Noncommutative algebra}_{\text{simplicial}}(\text{IndNormed}_?) / \text{Noncommutative algebra}_{\text{simplicial}}(\text{IndNormed}_?)^{\text{opposite}}, \text{Grothendieck topology, homotopy epimorphism} \quad (3.5.9)$$

$$\text{Proj}^{\text{smooth formal series closure}} \infty - \text{Toposes}_{\text{ringed, Noncommutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_?) / \text{Noncommutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_?)^{\text{opposite}}, \text{Grothendieck topology, homotopy epimorphism} \quad (3.5.10)$$

$$\text{Proj}^{\text{smooth formal series closure}} \infty - \text{Toposes}_{\text{ringed, Noncommutative algebra}_{\text{simplicial}}(\text{IndBanach}_?) / \text{Noncommutative algebra}_{\text{simplicial}}(\text{IndBanach}_?)^{\text{opposite}}, \text{Grothendieck topology, homotopy epimorphism} \quad (3.5.11)$$

$$\text{Proj}^{\text{smooth formal series closure}} \infty - \text{Toposes}_{\text{ringed, Noncommutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_?) / \text{Noncommutative algebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_?)^{\text{opposite}}, \text{Grothendieck topology, homotopy epimorphism} \quad (3.5.12)$$

$$(3.5.13)$$

Starting from the ∞ -rings, we have:

Definition 3.5.1. One can actually define the derived prismatic cohomologies through derived topological Hochschild cohomologies, derived topological period cohomologies and derived topological cyclic cohomologies as in [BMS, Section 2.2, Section 2.3], [BS1, Theorem 1.13]:

$$\text{Kan}_{\text{Left}} \text{THH}^{\text{noncommutative}}, \text{Kan}_{\text{Left}} \text{TP}^{\text{noncommutative}}, \text{Kan}_{\text{Left}} \text{TC}^{\text{noncommutative}}, \quad (3.5.14)$$

³⁰on the following $(\infty, 1)$ -compactly generated closures of the corresponding polynomials³¹ given over A/I with a chosen prism (A, I) ³²:

$$\text{Object}_{\mathbb{E}_{\infty}\text{Noncommutativealgebra, Simplicial}}(\text{IndSNorm}_{A/I})^{\text{smoothformalseriesclosure}}, \quad (3.5.15)$$

$$\text{Object}_{\mathbb{E}_{\infty}\text{Noncommutativealgebra, Simplicial}}(\text{Ind}^m\text{SNorm}_{A/I})^{\text{smoothformalseriesclosure}}, \quad (3.5.16)$$

$$\text{Object}_{\mathbb{E}_{\infty}\text{Noncommutativealgebra, Simplicial}}(\text{IndNorm}_{A/I})^{\text{smoothformalseriesclosure}}, \quad (3.5.17)$$

$$\text{Object}_{\mathbb{E}_{\infty}\text{Noncommutativealgebra, Simplicial}}(\text{Ind}^m\text{Norm}_{A/I})^{\text{smoothformalseriesclosure}}, \quad (3.5.18)$$

$$\text{Object}_{\mathbb{E}_{\infty}\text{Noncommutativealgebra, Simplicial}}(\text{IndBan}_{A/I})^{\text{smoothformalseriesclosure}}, \quad (3.5.19)$$

$$\text{Object}_{\mathbb{E}_{\infty}\text{Noncommutativealgebra, Simplicial}}(\text{Ind}^m\text{Ban}_{A/I})^{\text{smoothformalseriesclosure}}. \quad (3.5.20)$$

We call the corresponding functors are derived functional analytic Hochschild cohomologies, derived functional analytic period cohomologies and derived functional analytic cyclic cohomologies, which we are going to denote them as in the following:

$$\text{Kan}_{\text{Left}}\text{THH}_{\text{functionalanalytic, KKM, BBM, formalanalytification, nc}}^{\text{noncommutative}}(\mathcal{O}) := \quad (3.5.21)$$

$$\left(\text{homotopycolimit}_{i \text{ sifted, derivedcategory}_{\infty}(A/I\text{-Module})} \text{Kan}_{\text{Left}}\text{THH}_{\text{functionalanalytic, KKM}}^{\text{noncommutative}}(\mathcal{O}_i) \right) \quad (3.5.22)$$

$$\left. \right)_{\text{BBM, formalanalytification, nc}}, \quad (3.5.23)$$

$$\text{Kan}_{\text{Left}}\text{TP}_{\text{functionalanalytic, KKM, BBM, formalanalytification, nc}}^{\text{noncommutative}}(\mathcal{O}) := \quad (3.5.24)$$

$$\left(\text{homotopycolimit}_{i \text{ sifted, derivedcategory}_{\infty}(A/I\text{-Module})} \text{Kan}_{\text{Left}}\text{TP}_{\text{functionalanalytic, KKM}}^{\text{noncommutative}}(\mathcal{O}_i) \right) \quad (3.5.25)$$

$$\left. \right)_{\text{BBM, formalanalytification, nc}}, \quad (3.5.26)$$

$$\text{Kan}_{\text{Left}}\text{TC}_{\text{functionalanalytic, KKM, BBM, formalanalytification, nc}}^{\text{noncommutative}}(\mathcal{O}) := \quad (3.5.27)$$

$$\left(\text{homotopycolimit}_{i \text{ sifted, derivedcategory}_{\infty}(A/I\text{-Module})} \text{Kan}_{\text{Left}}\text{TC}_{\text{functionalanalytic, KKM}}^{\text{noncommutative}}(\mathcal{O}_i) \right) \quad (3.5.28)$$

$$\left. \right)_{\text{BBM, formalanalytification, nc}}, \quad (3.5.29)$$

by writing any object \mathcal{O} as the corresponding colimit

$$\text{homotopycolimit}_{i \text{ sifted}} \mathcal{O}_i.$$

These are quite large $(\infty, 1)$ -commutative ring objects in the corresponding $(\infty, 1)$ -categories for $R = A/I$:

³⁰One has the corresponding p -completed versions as well.

³¹Definitely, we need to put certain norms over in some relatively canonical way, as in [BBM, Section 4.2] one can basically consider rigid ones and dagger ones, and so on. In this noncommutative setting we do not actually need to fix the type of the analytification, as in commutative setting since we are going to apply directly construction from Bhatt-Scholze and Koshikawa [BS1] and [Ko1], though the derived characterization of the prismatic cohomology might not need to be restricted to the corresponding p -adic formal scheme situations.

³²In all the following, we assume this prism to be bounded and satisfy that A/I is Banach.

$$\text{Object}_{E_\infty \text{Noncommutativealgebra, Simplicial}}(\text{IndSNorm}_R), \quad (3.5.30)$$

$$\text{Object}_{E_\infty \text{Noncommutativealgebra, Simplicial}}(\text{Ind}^m \text{SNorm}_R), \quad (3.5.31)$$

$$\text{Object}_{E_\infty \text{Noncommutativealgebra, Simplicial}}(\text{IndNorm}_R), \quad (3.5.32)$$

$$\text{Object}_{E_\infty \text{Noncommutativealgebra, Simplicial}}(\text{Ind}^m \text{Norm}_R), \quad (3.5.33)$$

$$\text{Object}_{E_\infty \text{Noncommutativealgebra, Simplicial}}(\text{IndBan}_R), \quad (3.5.34)$$

$$\text{Object}_{E_\infty \text{Noncommutativealgebra, Simplicial}}(\text{Ind}^m \text{Ban}_R), \quad (3.5.35)$$

after taking the formal series ring left Kan extension analytification from [BBM, Section 4.2], which is defined by taking the left Kan extension to all the $(\infty, 1)$ -ring objects in the ∞ -derived category of all A -modules from formal series rings over A , into:

$$\text{Object}_{E_\infty \text{Noncommutativealgebra, Simplicial}}(\text{IndSNorm}_{\mathbb{F}_1})_A, \quad (3.5.36)$$

$$\text{Object}_{E_\infty \text{Noncommutativealgebra, Simplicial}}(\text{Ind}^m \text{SNorm}_{\mathbb{F}_1})_A, \quad (3.5.37)$$

$$\text{Object}_{E_\infty \text{Noncommutativealgebra, Simplicial}}(\text{IndNorm}_{\mathbb{F}_1})_A, \quad (3.5.38)$$

$$\text{Object}_{E_\infty \text{Noncommutativealgebra, Simplicial}}(\text{Ind}^m \text{Norm}_{\mathbb{F}_1})_A, \quad (3.5.39)$$

$$\text{Object}_{E_\infty \text{Noncommutativealgebra, Simplicial}}(\text{IndBan}_{\mathbb{F}_1})_A, \quad (3.5.40)$$

$$\text{Object}_{E_\infty \text{Noncommutativealgebra, Simplicial}}(\text{Ind}^m \text{Ban}_{\mathbb{F}_1})_A. \quad (3.5.41)$$

Definition 3.5.2. Then we can in the same fashion consider the corresponding derived prismatic complexes [BS1, Construction 7.6]³³ for the commutative algebras as in the above (for a given prism (A, I)):

$$\text{Kan}_{\text{Left}} \Delta_{?/A} := \text{Kan}_{\text{Left}} \text{TP}^{\text{noncommutative, pcomplete}}(?/A), \quad (3.5.42)$$

by the regular corresponding left Kan extension techniques on the following $(\infty, 1)$ -compactly generated closures of the corresponding polynomials given over A/I with a chosen prism (A, I) :

$$\text{Object}_{E_\infty \text{Noncommutativealgebra, Simplicial}}(\text{IndSNorm}_{A/I})^{\text{smoothformalseriesclosure}}, \quad (3.5.43)$$

$$\text{Object}_{E_\infty \text{Noncommutativealgebra, Simplicial}}(\text{Ind}^m \text{SNorm}_{A/I})^{\text{smoothformalseriesclosure}}, \quad (3.5.44)$$

$$\text{Object}_{E_\infty \text{Noncommutativealgebra, Simplicial}}(\text{IndNorm}_{A/I})^{\text{smoothformalseriesclosure}}, \quad (3.5.45)$$

$$\text{Object}_{E_\infty \text{Noncommutativealgebra, Simplicial}}(\text{Ind}^m \text{Norm}_{A/I})^{\text{smoothformalseriesclosure}}, \quad (3.5.46)$$

$$\text{Object}_{E_\infty \text{Noncommutativealgebra, Simplicial}}(\text{IndBan}_{A/I})^{\text{smoothformalseriesclosure}}, \quad (3.5.47)$$

$$\text{Object}_{E_\infty \text{Noncommutativealgebra, Simplicial}}(\text{Ind}^m \text{Ban}_{A/I})^{\text{smoothformalseriesclosure}}. \quad (3.5.48)$$

³³One just applies [BS1, Construction 7.6] and then takes the left Kan extensions.

We call the corresponding functors functional analytic derived prismatic complexes which we are going to denote that as in the following:

$$\mathbf{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, KKM, BBM, formalanalytification, nc}} \quad (3.5.49)$$

$$:= \mathbf{Kan}_{\text{Left}}\mathbf{TP}^{\text{noncommutative, pcomplete}}(?/A)_{\text{functionalanalytic, KKM, BBM, formalanalytification, nc}} \quad (3.5.50)$$

This would mean the following definition³⁴:

$$\mathbf{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, KKM, BBM, formalanalytification, nc}}(\mathcal{O}) \quad (3.5.51)$$

$$:= \left(\underset{i}{\text{homotopycolimit}}_{\text{sifted, derivedcategory}_{\infty}(A/I\text{-Module})} \mathbf{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, KKM}}(\mathcal{O}_i) \right)^{\wedge} \quad (3.5.52)$$

$$\text{)BBM, formalanalytification, nc} \quad (3.5.53)$$

$$= \left(\underset{i}{\text{homotopycolimit}}_{\text{sifted, derivedcategory}_{\infty}(A/I\text{-Module})} \mathbf{Kan}_{\text{Left}}\mathbf{TP}^{\text{noncommutative, pcomplete}} \quad (3.5.54)$$

$$(?/A)_{\text{functionalanalytic, KKM}}(\mathcal{O}_i) \right)^{\wedge} \text{)BBM, formalanalytification, nc} \quad (3.5.55)$$

$$(3.5.56)$$

by writing any object \mathcal{O} as the corresponding colimit

$$\underset{i}{\text{homotopycolimit}}_{\text{sifted}} \mathcal{O}_i.$$

These are quite large $(\infty, 1)$ -commutative ring objects in the corresponding $(\infty, 1)$ -categories for $R = A/I$:

$$\mathbf{Object}_{\mathbf{E}_{\infty}\text{Noncommutativealgebra, Simplicial}}(\mathbf{IndSNorm}_R), \quad (3.5.57)$$

$$\mathbf{Object}_{\mathbf{E}_{\infty}\text{Noncommutativealgebra, Simplicial}}(\mathbf{Ind}^m\mathbf{SNorm}_R), \quad (3.5.58)$$

$$\mathbf{Object}_{\mathbf{E}_{\infty}\text{Noncommutativealgebra, Simplicial}}(\mathbf{IndNorm}_R), \quad (3.5.59)$$

$$\mathbf{Object}_{\mathbf{E}_{\infty}\text{Noncommutativealgebra, Simplicial}}(\mathbf{Ind}^m\mathbf{Norm}_R), \quad (3.5.60)$$

$$\mathbf{Object}_{\mathbf{E}_{\infty}\text{Noncommutativealgebra, Simplicial}}(\mathbf{IndBan}_R), \quad (3.5.61)$$

$$\mathbf{Object}_{\mathbf{E}_{\infty}\text{Noncommutativealgebra, Simplicial}}(\mathbf{Ind}^m\mathbf{Ban}_R), \quad (3.5.62)$$

$$(3.5.63)$$

after taking the formal series ring left Kan extension analytification from [BBM, Section 4.2], which is defined by taking the left Kan extension to all the $(\infty, 1)$ -ring objects in the ∞ -derived

³⁴Before the Ben-Bassat-Mukherjee p -adic formal analytification we take the corresponding p -completion.

category of all A -modules from formal series rings over A , into:

$$\text{Object}_{E_\infty\text{Noncommutativealgebra,Simplicial}}(\text{IndSNorm}_{\mathbb{F}_1})_A, \quad (3.5.64)$$

$$\text{Object}_{E_\infty\text{Noncommutativealgebra,Simplicial}}(\text{Ind}^m\text{SNorm}_{\mathbb{F}_1})_A, \quad (3.5.65)$$

$$\text{Object}_{E_\infty\text{Noncommutativealgebra,Simplicial}}(\text{IndNorm}_{\mathbb{F}_1})_A, \quad (3.5.66)$$

$$\text{Object}_{E_\infty\text{Noncommutativealgebra,Simplicial}}(\text{Ind}^m\text{Norm}_{\mathbb{F}_1})_A, \quad (3.5.67)$$

$$\text{Object}_{E_\infty\text{Noncommutativealgebra,Simplicial}}(\text{IndBan}_{\mathbb{F}_1})_A, \quad (3.5.68)$$

$$\text{Object}_{E_\infty\text{Noncommutativealgebra,Simplicial}}(\text{Ind}^m\text{Ban}_{\mathbb{F}_1})_A. \quad (3.5.69)$$

Then as in [BS1, Definition 8.2] we consider the corresponding ‘perfectoidization’ in this analytic setting.

Definition 3.5.3. Let (A, I) be a perfectoid prism, and we consider any E_∞ -ring \mathcal{O} in the following

$$\text{Object}_{E_\infty\text{Noncommutativealgebra,Simplicial}}(\text{IndSNorm}_{A/I})^{\text{smoothformalseriesclosure}}, \quad (3.5.70)$$

$$\text{Object}_{E_\infty\text{Noncommutativealgebra,Simplicial}}(\text{Ind}^m\text{SNorm}_{A/I})^{\text{smoothformalseriesclosure}}, \quad (3.5.71)$$

$$\text{Object}_{E_\infty\text{Noncommutativealgebra,Simplicial}}(\text{IndNorm}_{A/I})^{\text{smoothformalseriesclosure}}, \quad (3.5.72)$$

$$\text{Object}_{E_\infty\text{Noncommutativealgebra,Simplicial}}(\text{Ind}^m\text{Norm}_{A/I})^{\text{smoothformalseriesclosure}}, \quad (3.5.73)$$

$$\text{Object}_{E_\infty\text{Noncommutativealgebra,Simplicial}}(\text{IndBan}_{A/I})^{\text{smoothformalseriesclosure}}, \quad (3.5.74)$$

$$\text{Object}_{E_\infty\text{Noncommutativealgebra,Simplicial}}(\text{Ind}^m\text{Ban}_{A/I})^{\text{smoothformalseriesclosure}}. \quad (3.5.75)$$

Then consider the derived prismatic object:

$$\text{Kan}_{\text{Left}}\Delta_{?/A,\text{functionalanalytic,KKM,BBM,formalanalytification,nc}}(\mathcal{O}) \quad (3.5.76)$$

$$:= \text{Kan}_{\text{Left}}\text{TP}^{\text{noncommutative,pcomplete}}(_{?/A})_{\text{functionalanalytic,KKM,BBM,formalanalytification,nc}}(\mathcal{O}). \quad (3.5.77)$$

Then as in [BS1, Definition 8.2] we have the following ‘preperfectoidization’:

$$(\mathcal{O})^{\text{preperfectoidization}} \quad (3.5.78)$$

$$:= \text{Colimit}(\text{Kan}_{\text{Left}}\Delta_{?/A,\text{functionalanalytic,KKM,BBM,formalanalytification,nc}}(\mathcal{O}) \rightarrow \quad (3.5.79)$$

$$\phi_*\text{Kan}_{\text{Left}}\Delta_{?/A,\text{functionalanalytic,KKM,BBM,formalanalytification,nc}}(\mathcal{O}) \quad (3.5.80)$$

$$\rightarrow \phi_*\phi_*\text{Kan}_{\text{Left}}\Delta_{?/A,\text{functionalanalytic,KKM,BBM,formalanalytification,nc}}(\mathcal{O}) \rightarrow \dots)_{\text{BBM,formalanalytification,nc}}, \quad (3.5.81)$$

after taking the formal series ring left Kan extension analytification from [BBM, Section 4.2], which is defined by taking the left Kan extension to all the $(\infty, 1)$ -ring object in the ∞ -derived category of all A -modules from formal series rings over A . Then we define the corresponding

'perfectoidization':

$$(\mathcal{O})^{\text{perfectoidization}} \quad (3.5.82)$$

$$:= \text{Colimit}(\text{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, KKM, BBM, formalanalytification, nc}}(\mathcal{O}) \longrightarrow \quad (3.5.83)$$

$$\phi_* \text{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, KKM, BBM, formalanalytification, nc}}(\mathcal{O}) \quad (3.5.84)$$

$$\longrightarrow \phi_* \phi_* \text{Kan}_{\text{Left}}\Delta_{?/A, \text{functionalanalytic, KKM, BBM, formalanalytification, nc}}(\mathcal{O}) \longrightarrow \dots \quad (3.5.85)$$

$$\text{BBM, formalanalytification, nc} \times A/I. \quad (3.5.86)$$

Furthermore one can take derived (p, I) -completion to achieve the derived (p, I) -completed versions:

$$\mathcal{O}^{\text{preperfectoidization, derivedcomplete}} := (\mathcal{O}^{\text{preperfectoidization}})^\wedge, \quad (3.5.87)$$

$$\mathcal{O}^{\text{perfectoidization, derivedcomplete}} := \mathcal{O}^{\text{preperfectoidization, derivedcomplete}} \times A/I. \quad (3.5.88)$$

$$(3.5.89)$$

These are large $(\infty, 1)$ -noncommutative algebra objects in the corresponding categories as in the above, attached to also large $(\infty, 1)$ -noncommutative algebra objects. When we apply this to the corresponding sub- $(\infty, 1)$ -categories of Banach perfectoid objects as in [BMS2], [GR], [KL1], [KL2], [Ked1], [Sch3], we will recover the corresponding noncommutative analogues of the distinguished elemental deformation processes defined in [BMS2], [GR], [KL1], [KL2], [Ked1], [Sch3].

Remark 3.5.4. One can then define such ring \mathcal{O} to be *preperfectoid* if we have the equivalence:

$$\mathcal{O}^{\text{preperfectoidization}} \xrightarrow{\sim} \mathcal{O}. \quad (3.5.90)$$

One can then define such ring \mathcal{O} to be *perfectoid* if we have the equivalence:

$$\mathcal{O}^{\text{preperfectoidization}} \times A/I \xrightarrow{\sim} \mathcal{O}. \quad (3.5.91)$$

Definition 3.5.5. One can actually define the derived prismatic cohomology presheaves through derived topological Hochschild cohomology presheaves, derived topological period cohomology presheaves and derived topological cyclic cohomology presheaves as in [BMS, Section 2.2, Section 2.3], [BS1, Theorem 1.13]:

$$\text{Kan}_{\text{Left}}\text{THH}^{\text{noncommutative}}, \text{Kan}_{\text{Left}}\text{TP}^{\text{noncommutative}}, \text{Kan}_{\text{Left}}\text{TC}^{\text{noncommutative}}, \quad (3.5.92)$$

on the following $(\infty, 1)$ -compactly generated closures of the corresponding polynomials³⁵ given

³⁵Definitely, we need to put certain norms over in some relatively canonical way, as in [BBM, Section 4.2] one can basically consider rigid ones and dagger ones, and so on. Again in this noncommutative setting we do not actually need to fix the type of the analytification, as in commutative setting since we are going to apply directly construction from Bhatt-Scholze and Koshikawa [BS1] and [Ko1], though the derived characterization of the prismatic cohomology might not need to be restricted to the corresponding p -adic formal scheme situations.

over A/I with a chosen prism (A, I) ³⁶:

$$\text{Proj}^{\text{smoothformalseriesclosure}}_{\infty} - \text{Toposes}_{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_{A/I})}^{\text{Noncommutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_{A/I})^{\text{opposite}}, \text{Grothendiecktopology, homotopyepim}} \quad (3.5.93)$$

$$\text{Proj}^{\text{smoothformalseriesclosure}}_{\infty} - \text{Toposes}_{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_{A/I})}^{\text{Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_{A/I})^{\text{opposite}}, \text{Grothendiecktopology, homotopyepim}} \quad (3.5.94)$$

$$\text{Proj}^{\text{smoothformalseriesclosure}}_{\infty} - \text{Toposes}_{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{IndNormed}_{A/I})}^{\text{Noncommutativealgebra}_{\text{simplicial}}(\text{IndNormed}_{A/I})^{\text{opposite}}, \text{Grothendiecktopology, homotopyepim}} \quad (3.5.95)$$

$$\text{Proj}^{\text{smoothformalseriesclosure}}_{\infty} - \text{Toposes}_{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_{A/I})}^{\text{Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_{A/I})^{\text{opposite}}, \text{Grothendiecktopology, homotopyepim}} \quad (3.5.96)$$

$$\text{Proj}^{\text{smoothformalseriesclosure}}_{\infty} - \text{Toposes}_{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{IndBanach}_{A/I})}^{\text{Noncommutativealgebra}_{\text{simplicial}}(\text{IndBanach}_{A/I})^{\text{opposite}}, \text{Grothendiecktopology, homotopyepim}} \quad (3.5.97)$$

$$\text{Proj}^{\text{smoothformalseriesclosure}}_{\infty} - \text{Toposes}_{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_{A/I})}^{\text{Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_{A/I})^{\text{opposite}}, \text{Grothendiecktopology, homotopyepim}} \quad (3.5.98)$$

We call the corresponding functors are derived functional analytic Hochschild cohomology presheaves, derived functional analytic period cohomology presheaves and derived functional analytic cyclic cohomology presheaves, which we are going to denote these presheaves as in the following for any ∞ -ringed topos $(\mathbb{X}, \mathcal{O}) = \text{homotopycolimit}_i(\mathbb{X}_i, \mathcal{O}_i)$:

$$\text{Kan}_{\text{Left}} \text{THH}_{\text{functionalanalytic, KKM, BBM, formalanalytification, nc}}^{\text{noncommutative}}(\mathcal{O}) := \quad (3.5.99)$$

$$(\text{homotopycolimit}_i \text{ sifted, derivedcategory}_{\infty}(A/I\text{-Module}) \text{Kan}_{\text{Left}} \text{THH}_{\text{functionalanalytic, KKM}}^{\text{noncommutative}}(\mathcal{O}_i) \quad (3.5.100)$$

$$) \text{BBM, formalanalytification, nc} \quad (3.5.101)$$

$$\text{Kan}_{\text{Left}} \text{TP}_{\text{functionalanalytic, KKM, BBM, formalanalytification, nc}}^{\text{noncommutative}}(\mathcal{O}) := \quad (3.5.102)$$

$$(\text{homotopycolimit}_i \text{ sifted, derivedcategory}_{\infty}(A/I\text{-Module}) \text{Kan}_{\text{Left}} \text{TP}_{\text{functionalanalytic, KKM}}^{\text{noncommutative}}(\mathcal{O}_i) \quad (3.5.103)$$

$$) \text{BBM, formalanalytification, nc} \quad (3.5.104)$$

$$\text{Kan}_{\text{Left}} \text{TC}_{\text{functionalanalytic, KKM, BBM, formalanalytification, nc}}^{\text{noncommutative}}(\mathcal{O}) := \quad (3.5.105)$$

$$(\text{homotopycolimit}_i \text{ sifted, derivedcategory}_{\infty}(A/I\text{-Module}) \text{Kan}_{\text{Left}} \text{TC}_{\text{functionalanalytic, KKM}}^{\text{noncommutative}}(\mathcal{O}_i) \quad (3.5.106)$$

$$) \text{BBM, formalanalytification, nc} \quad (3.5.107)$$

by writing any object \mathcal{O} as the corresponding colimit³⁷

³⁶In all the following, we assume this prism to be bounded and satisfy that A/I is Banach.

³⁷Here we assume that in the following the presheaves \mathcal{O}_i are taking values in the derived p -completed \mathbb{E}_1 -algebras over A/I , which then are generated by those p -adic formal series ring $A/I\langle Z_1, \dots, Z_n \rangle$, $n = 1, 2, \dots$ by the corresponding derived colimit completion with free variables Z_1, \dots, Z_n , $n = 1, 2, \dots$

$$\operatorname{homotopycolimit}_{i \text{ sifted}} \mathcal{O}_i.$$

These are quite large $(\infty, 1)$ -commutative ring objects in the corresponding $(\infty, 1)$ -categories for $? = A/I$ over $(\mathbb{X}, \mathcal{O})$:

$$\operatorname{Ind}\# \operatorname{Quasicoherent}^{\operatorname{presheaf}}_{\operatorname{Ind}^{\operatorname{smoothformalseriesclosure}} \infty\text{-Toposes}} \begin{array}{l} \operatorname{ringed, Noncommutativealgebra}_{\operatorname{simplicial}}(\operatorname{IndSeminormed}_?) \\ \operatorname{Noncommutativealgebra}_{\operatorname{simplicial}}(\operatorname{IndSeminormed}_?)^{\operatorname{opposite}}, \operatorname{Grotopology, homotopyepimorphism} \end{array} \quad (3.5.108)$$

$$\operatorname{Ind}\# \operatorname{Quasicoherent}^{\operatorname{presheaf}}_{\operatorname{Ind}^{\operatorname{smoothformalseriesclosure}} \infty\text{-Toposes}} \begin{array}{l} \operatorname{ringed, Noncommutativealgebra}_{\operatorname{simplicial}}(\operatorname{Ind}^m \operatorname{Seminormed}_?) \\ \operatorname{Noncommutativealgebra}_{\operatorname{simplicial}}(\operatorname{Ind}^m \operatorname{Seminormed}_?)^{\operatorname{opposite}}, \operatorname{Grotopology, homotopyepimorphism} \end{array} \quad (3.5.109)$$

$$\operatorname{Ind}\# \operatorname{Quasicoherent}^{\operatorname{presheaf}}_{\operatorname{Ind}^{\operatorname{smoothformalseriesclosure}} \infty\text{-Toposes}} \begin{array}{l} \operatorname{ringed, Noncommutativealgebra}_{\operatorname{simplicial}}(\operatorname{IndNormed}_?) \\ \operatorname{Noncommutativealgebra}_{\operatorname{simplicial}}(\operatorname{IndNormed}_?)^{\operatorname{opposite}}, \operatorname{Grotopology, homotopyepimorphism} \end{array} \quad (3.5.110)$$

$$\operatorname{Ind}\# \operatorname{Quasicoherent}^{\operatorname{presheaf}}_{\operatorname{Ind}^{\operatorname{smoothformalseriesclosure}} \infty\text{-Toposes}} \begin{array}{l} \operatorname{ringed, Noncommutativealgebra}_{\operatorname{simplicial}}(\operatorname{Ind}^m \operatorname{Normed}_?) \\ \operatorname{Noncommutativealgebra}_{\operatorname{simplicial}}(\operatorname{Ind}^m \operatorname{Normed}_?)^{\operatorname{opposite}}, \operatorname{Grotopology, homotopyepimorphism} \end{array} \quad (3.5.111)$$

$$\operatorname{Ind}\# \operatorname{Quasicoherent}^{\operatorname{presheaf}}_{\operatorname{Ind}^{\operatorname{smoothformalseriesclosure}} \infty\text{-Toposes}} \begin{array}{l} \operatorname{ringed, Noncommutativealgebra}_{\operatorname{simplicial}}(\operatorname{IndBanach}_?) \\ \operatorname{Noncommutativealgebra}_{\operatorname{simplicial}}(\operatorname{IndBanach}_?)^{\operatorname{opposite}}, \operatorname{Grotopology, homotopyepimorphism} \end{array} \quad (3.5.112)$$

$$\operatorname{Ind}\# \operatorname{Quasicoherent}^{\operatorname{presheaf}}_{\operatorname{Ind}^{\operatorname{smoothformalseriesclosure}} \infty\text{-Toposes}} \begin{array}{l} \operatorname{ringed, Noncommutativealgebra}_{\operatorname{simplicial}}(\operatorname{Ind}^m \operatorname{Banach}_?) \\ \operatorname{Noncommutativealgebra}_{\operatorname{simplicial}}(\operatorname{Ind}^m \operatorname{Banach}_?)^{\operatorname{opposite}}, \operatorname{Grotopology, homotopyepimorphism} \end{array} \quad (3.5.113)$$

$$(3.5.114)$$

after taking the formal series ring left Kan extension analytification from [BBM, Section 4.2], which is defined by taking the left Kan extension to all the $(\infty, 1)$ -ring objects in the ∞ -derived category of all A -modules from formal series rings over A .

Assumption 3.5.6. (Technical Assumption) Here we assume that in the following the presheaves \mathcal{O}_i are taking values in the derived p -completed \mathbb{E}_1 -algebras over A/I , which then are generated by those p -adic formal series ring $A/I \langle Z_1, \dots, Z_n \rangle$, $n = 1, 2, \dots$ by the corresponding derived colimit completion with free variables Z_1, \dots, Z_n , $n = 1, 2, \dots$. However this assumption does not really matter so significantly in this noncommutative situation since we are just taking the direct definition through TP.

Definition 3.5.7. Then we can in the same fashion consider the corresponding derived noncommutative prismatic complex presheaves [BS1, Construction 7.6]³⁸ for the commutative algebras as in the above (for a given prism (A, I)):

$$\operatorname{Kan}_{\operatorname{Left}} \Delta_{?/A} := \operatorname{Kan}_{\operatorname{Left}} \operatorname{TP}^{\operatorname{noncommutative, pcomplete}}(?/A), \quad (3.5.115)$$

³⁸One just applies [BS1, Construction 7.6] and then takes the left Kan extensions.

³⁹by the regular corresponding left Kan extension techniques on the following $(\infty, 1)$ -compactly generated closures of the corresponding polynomials given over A/I with a chosen prism (A, I) :

$$\text{Proj}^{\text{smoothformalseriesclosure}}_{\infty} - \text{Toposes}_{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_{A/I}) / \text{Noncommutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_{A/I})^{\text{opposite}}, \text{Grothendiecktopology, homotopyepim}} \quad (3.5.116)$$

$$\text{Proj}^{\text{smoothformalseriesclosure}}_{\infty} - \text{Toposes}_{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_{A/I}) / \text{Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Seminormed}_{A/I})^{\text{opposite}}, \text{Grothendiecktopology, homotopyepim}} \quad (3.5.117)$$

$$\text{Proj}^{\text{smoothformalseriesclosure}}_{\infty} - \text{Toposes}_{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{IndNormed}_{A/I}) / \text{Noncommutativealgebra}_{\text{simplicial}}(\text{IndNormed}_{A/I})^{\text{opposite}}, \text{Grothendiecktopology, homotopyepim}} \quad (3.5.118)$$

$$\text{Proj}^{\text{smoothformalseriesclosure}}_{\infty} - \text{Toposes}_{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_{A/I}) / \text{Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Normed}_{A/I})^{\text{opposite}}, \text{Grothendiecktopology, homotopyepim}} \quad (3.5.119)$$

$$\text{Proj}^{\text{smoothformalseriesclosure}}_{\infty} - \text{Toposes}_{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{IndBanach}_{A/I}) / \text{Noncommutativealgebra}_{\text{simplicial}}(\text{IndBanach}_{A/I})^{\text{opposite}}, \text{Grothendiecktopology, homotopyepim}} \quad (3.5.120)$$

$$\text{Proj}^{\text{smoothformalseriesclosure}}_{\infty} - \text{Toposes}_{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_{A/I}) / \text{Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m \text{Banach}_{A/I})^{\text{opposite}}, \text{Grothendiecktopology, homotopyepim}} \quad (3.5.121)$$

We call the corresponding functors functional analytic derived prismatic complex presheaves which we are going to denote that as in the following:

$$\text{Kan}_{\text{Left}} \Delta_{?/A, \text{functionalanalytic, KKM, BBM, formalanalytification, nc}} \quad (3.5.122)$$

$$:= \text{Kan}_{\text{Left}} \text{TP}^{\text{noncommutative, pcomplete}}(? / A)_{\text{functionalanalytic, KKM, BBM, formalanalytification, nc}} \quad (3.5.123)$$

This would mean the following definition⁴⁰:

$$\text{Kan}_{\text{Left}} \Delta_{?/A, \text{functionalanalytic, KKM, BBM, formalanalytification, nc}}(\mathcal{O}) \quad (3.5.124)$$

$$:= \left(\underset{i}{\text{homotopycolimit}} \underset{\text{sifted, derivedcategory}_{\infty}(A/I\text{-Module})}{\text{Kan}_{\text{Left}} \Delta_{?/A, \text{functionalanalytic, KKM}}(\mathcal{O}_i)} \right)^{\wedge} \quad (3.5.125)$$

$$\text{BBM, formalanalytification, nc} \quad (3.5.126)$$

by writing any object \mathcal{O} as the corresponding colimit

$$\underset{i}{\text{homotopycolimit}} \underset{\text{sifted}}{\mathcal{O}_i}$$

³⁹The motivation for this definition comes from the corresponding commutative picture, namely the corresponding topological characterization of the corresponding prismatic cohomology and the corresponding completed version by using the corresponding Nygaard filtrations. See [BS1, Theorem 1.13].

⁴⁰Before the Ben-Bassat-Mukherjee p -adic formal analytification we take the corresponding p -completion.

These are quite large $(\infty, 1)$ -commutative ring objects in the corresponding $(\infty, 1)$ -categories for $R = A/I$:

$$\text{Ind}\#_{\text{Quasicoherent}}^{\text{presheaf}} \underset{\text{Ind}^{\text{smoothformalseriesclosure}}}{\infty\text{-Toposes}} \underset{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_R)}{\text{Noncommutativealgebra}_{\text{simplicial}}(\text{IndSeminormed}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}} \cdot$$

(3.5.127)

$$\text{Ind}\#_{\text{Quasicoherent}}^{\text{presheaf}} \underset{\text{Ind}^{\text{smoothformalseriesclosure}}}{\infty\text{-Toposes}} \underset{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m\text{Seminormed}_R)}{\text{Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m\text{Seminormed}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}} \cdot$$

(3.5.128)

$$\text{Ind}\#_{\text{Quasicoherent}}^{\text{presheaf}} \underset{\text{Ind}^{\text{smoothformalseriesclosure}}}{\infty\text{-Toposes}} \underset{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{IndNormed}_R)}{\text{Noncommutativealgebra}_{\text{simplicial}}(\text{IndNormed}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}} \cdot$$

(3.5.129)

$$\text{Ind}\#_{\text{Quasicoherent}}^{\text{presheaf}} \underset{\text{Ind}^{\text{smoothformalseriesclosure}}}{\infty\text{-Toposes}} \underset{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m\text{Normed}_R)}{\text{Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m\text{Normed}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}} \cdot$$

(3.5.130)

$$\text{Ind}\#_{\text{Quasicoherent}}^{\text{presheaf}} \underset{\text{Ind}^{\text{smoothformalseriesclosure}}}{\infty\text{-Toposes}} \underset{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{IndBanach}_R)}{\text{Noncommutativealgebra}_{\text{simplicial}}(\text{IndBanach}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}} \cdot$$

(3.5.131)

$$\text{Ind}\#_{\text{Quasicoherent}}^{\text{presheaf}} \underset{\text{Ind}^{\text{smoothformalseriesclosure}}}{\infty\text{-Toposes}} \underset{\text{ringed, Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m\text{Banach}_R)}{\text{Noncommutativealgebra}_{\text{simplicial}}(\text{Ind}^m\text{Banach}_R)^{\text{opposite}}, \text{Grotopology, homotopyepimorphism}} \cdot$$

(3.5.132)

(3.5.133)

after taking the formal series ring left Kan extension analytification from [BBM, Section 4.2], which is defined by taking the left Kan extension to all the $(\infty, 1)$ -ring objects in the ∞ -derived category of all A -modules from formal series rings over A .

Then as in [BS1, Definition 8.2] we consider the corresponding noncommutative perfectoidization in this analytic setting.

Definition 3.5.8. Let (A, I) be a perfectoid prism, and we consider any E_∞ -ring \mathcal{O} in the fol-

lowing

$$\mathrm{Proj}^{\mathrm{smoothformalseriesclosure}_\infty} - \mathrm{Toposes}_{\mathrm{ringed,Noncommutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndSeminormed}_{A/I}), \mathrm{Noncommutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndSeminormed}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology}, \mathrm{homotopyepim}} \quad (3.5.134)$$

$$\mathrm{Proj}^{\mathrm{smoothformalseriesclosure}_\infty} - \mathrm{Toposes}_{\mathrm{ringed,Noncommutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Seminormed}_{A/I}), \mathrm{Noncommutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Seminormed}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology}, \mathrm{homotopyepim}} \quad (3.5.135)$$

$$\mathrm{Proj}^{\mathrm{smoothformalseriesclosure}_\infty} - \mathrm{Toposes}_{\mathrm{ringed,Noncommutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndNormed}_{A/I}), \mathrm{Noncommutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndNormed}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology}, \mathrm{homotopyepim}} \quad (3.5.136)$$

$$\mathrm{Proj}^{\mathrm{smoothformalseriesclosure}_\infty} - \mathrm{Toposes}_{\mathrm{ringed,Noncommutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Normed}_{A/I}), \mathrm{Noncommutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Normed}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology}, \mathrm{homotopyepim}} \quad (3.5.137)$$

$$\mathrm{Proj}^{\mathrm{smoothformalseriesclosure}_\infty} - \mathrm{Toposes}_{\mathrm{ringed,Noncommutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndBanach}_{A/I}), \mathrm{Noncommutativealgebra}_{\mathrm{simplicial}}(\mathrm{IndBanach}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology}, \mathrm{homotopyepim}} \quad (3.5.138)$$

$$\mathrm{Proj}^{\mathrm{smoothformalseriesclosure}_\infty} - \mathrm{Toposes}_{\mathrm{ringed,Noncommutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Banach}_{A/I}), \mathrm{Noncommutativealgebra}_{\mathrm{simplicial}}(\mathrm{Ind}^m \mathrm{Banach}_{A/I})^{\mathrm{opposite}}, \mathrm{Grothendiecktopology}, \mathrm{homotopyepim}} \quad (3.5.139)$$

Then consider the derived prismatic object:

$$\mathrm{Kan}_{\mathrm{Left}\Delta_{?/A}, \mathrm{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \mathrm{formalanalytification}, \mathrm{nc}}(\mathcal{O}). \quad (3.5.140)$$

Then as in [BS1, Definition 8.2] we have the following preperfectoidization:

$$(\mathcal{O})^{\mathrm{preperfectoidization}} \quad (3.5.141)$$

$$:= \mathrm{Colimit}(\mathrm{Kan}_{\mathrm{Left}\Delta_{?/A}, \mathrm{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \mathrm{formalanalytification}, \mathrm{nc}}(\mathcal{O}) \rightarrow \quad (3.5.142)$$

$$\phi_* \mathrm{Kan}_{\mathrm{Left}\Delta_{?/A}, \mathrm{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \mathrm{formalanalytification}, \mathrm{nc}}(\mathcal{O}) \quad (3.5.143)$$

$$\rightarrow \phi_* \phi_* \mathrm{Kan}_{\mathrm{Left}\Delta_{?/A}, \mathrm{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \mathrm{formalanalytification}, \mathrm{nc}}(\mathcal{O}) \rightarrow \dots)^{\mathrm{BBM}, \mathrm{formalanalytification}, \mathrm{nc}}, \quad (3.5.144)$$

after taking the formal series ring left Kan extension analytification from [BBM, Section 4.2], which is defined by taking the left Kan extension to all the $(\infty, 1)$ -ring object in the ∞ -derived category of all A -modules from formal series rings over A . Then we define the corresponding perfectoidization:

$$(\mathcal{O})^{\mathrm{perfectoidization}} \quad (3.5.145)$$

$$:= \mathrm{Colimit}(\mathrm{Kan}_{\mathrm{Left}\Delta_{?/A}, \mathrm{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \mathrm{formalanalytification}, \mathrm{nc}}(\mathcal{O}) \longrightarrow \quad (3.5.146)$$

$$\phi_* \mathrm{Kan}_{\mathrm{Left}\Delta_{?/A}, \mathrm{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \mathrm{formalanalytification}, \mathrm{nc}}(\mathcal{O}) \quad (3.5.147)$$

$$\longrightarrow \phi_* \phi_* \mathrm{Kan}_{\mathrm{Left}\Delta_{?/A}, \mathrm{functionalanalytic}, \mathrm{KKM}, \mathrm{BBM}, \mathrm{formalanalytification}, \mathrm{nc}}(\mathcal{O}) \longrightarrow \dots)^{\mathrm{BBM}, \mathrm{formalanalytification}, \mathrm{nc}} \times A/I. \quad (3.5.148)$$

Furthermore one can take derived (p, I) -completion to achieve the derived (p, I) -completed

versions:

$$\mathcal{O}^{\text{preperfectoidization, derivedcomplete}} := (\mathcal{O}^{\text{preperfectoidization}})^\wedge, \quad (3.5.149)$$

$$\mathcal{O}^{\text{perfectoidization, derivedcomplete}} := \mathcal{O}^{\text{preperfectoidization, derivedcomplete}} \times A/I. \quad (3.5.150)$$

$$(3.5.151)$$

These are large $(\infty, 1)$ -commutative algebra objects in the corresponding categories as in the above, attached to also large $(\infty, 1)$ -commutative algebra objects. When we apply this to the corresponding sub- $(\infty, 1)$ -categories of Banach perfectoid objects in [BMS2], [GR], [KL1], [KL2], [Ked1], [Sch3], we will have the corresponding noncommutative analogues of the distinguished elemental deformation processes defined in [BMS2], [GR], [KL1], [KL2], [Ked1], [Sch3].

Remark 3.5.9. One can then define such ring \mathcal{O} to be *preperfectoid* if we have the equivalence:

$$\mathcal{O}^{\text{preperfectoidization}} \xrightarrow{\sim} \mathcal{O}. \quad (3.5.152)$$

One can then define such ring \mathcal{O} to be *perfectoid* if we have the equivalence:

$$\mathcal{O}^{\text{preperfectoidization}} \times A/I \xrightarrow{\sim} \mathcal{O}. \quad (3.5.153)$$

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