∞-Categorical Approaches to Hodge-Iwasawa Theory II: ∞-Categorical and Derived Hodge-Iwasawa Modules

Xin Tong

Abstract

This paper is a discussion on ∞-categorical approaches to Hodge-Iwasawa Theory, which was initiated in our project on the ∞-categorical approaches to Hodge-Iwasawa Theory. The theory aims at the serious unification of p-adic Hodge Theory and p-adic Iwasawa Theory, by taking deformation of Hodge-theoretic constructions along some consideration in Iwasawa Theory beyond the Iwasawa deformation of certain motives in the general sense. The Hodge modules in our current consideration will be essentially within ∞-categorical derived categories of inductive Banach modules and ∞-categorical derived categories of condensed solidification of certain topological modules.

Contents

Chapter 1

Introduction

1.1 Results and Notations

1.1.1 Introduction to the Main Ideas

We discuss in this article the corresponding ∞-categorical and homotopicalization of some our serious consideration in [T2], with some goal in mind to generalize [KL1] and [KL2] along some deformation point of view. We hope to remind the readers of the fact that Hodge-Iwasawa theory is a combination of consideration in both Hodge theory and Iwasawa theory with certain moduli stack consideration closely after [BF1], [BF2], [FK], [FS], [He], [KL1], [KL2], [KP], [KPX], [Na1], [Na2], [PR], [RZ], [Sch2], [SW], [Wit]. The moduli stack consideration is following those in [He1-2], [HH-2], [HV-2], [PR-2], [RZ-2], [SW-2], [FS-2], [Sch-2], [Laff-2], [GL-2], [Ked-2], [Har1-2], [Har2-2], [EG-2], [EGH], [HHS], [D], [L], [W][1]. We are also inspired by [AB1-3], [AB2-3], [AB3-3], [AI1-3], [AI2-3], [AI3-3], [BMS1-3], [BMS2-3], [BS1-3], [BS2-3], [Fa1-3], [Fa2-3], [Fa3-3], [Fon1-3], [Fon2-3], [Fon3-3], [Fon4-3], [Fon5-3], [Iwa-3], [Ka1-3], [Ka2-3], [KL1-3], [KL2-3], [KL3-3], [KL4-3], [KP-3], [Lu1-3], [Lu2-3], [Lu3-3], [Mann-3], [Sch1-3], [Sch3-3], [Wi1-3], [Wi2-3], [Wi3-3], [HP]. We consider the multivariatization as well from [CKZ], [PZ] and [BCM]. The deformation is so important that we need to establish the corresponding results even in the situation where the spaces are relatively complicated to study. For instance the importance to look at ∞-categorical and homotopical aspects took the deep roots from the sheafiness of the topological period rings we are considering. However the sheafiness issue could be weakened after we look at some development from [BK], [BBBK], [BBK], [KKM], [BBM], [CS1], [CS2]. Generalizing the sheafiness to derived sheafiness, one could even achieve more such as the corresponding ∞-descent in quite general sense. The following two examples of results give the main ideas of the picture along the discussion above after [BK], [BBBK], [BBK], [KKM], [BBM], [CS1], [CS2].

Proposition 1.1.1. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Condensed}_{*} \tag{1.1.1}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.1.2}$$

$$\tag{1.1.3}$$

[1]The consideration will be essentially after [EG-2], [EGH], [HHS]. See [EGH, Conjecture 5.1.18, Section 5.2, Theorem 5.2.4] for the detail of Emerton-Gee-Hellmann conjecture on the moduli stack of (φ,Γ)-modules.

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.1.4}$$

$$\tag{1.1.5}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.1.6}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,A}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{1.1.7}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{1.1.8}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I[2].

[2] This means that we can descend to some ring or space with respect to some radius r, since we are talking about the corresponding projective limits of stacks (carrying inductive limits of ∞-sheaves of rings). This is in some sense (especially when we consider finally the corresponding possibly very non-quasicompact space X instead of a ring A) deviating from the situations above over full Robba rings where when deforming over X for instance noncompact we will achieve families of the parameters r going to ∞. This is not actually hard to understand since we are taking two combined limits in different orders:

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{Y\subset X}{\mathrm{homotopycolimit}} \neq \underset{Y\subset X}{\mathrm{homotopycolimit}}\ \underset{r}{\mathrm{homotopycolimit}}. \tag{1.1.9}$$

Proposition 1.1.2. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{IndBanach}_*. \tag{1.1.10}$$

Proposition 1.1.3. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves}, \text{Perfectcomplex}, \text{Condensed}_* \tag{1.1.11}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{CS}}\widetilde{\Phi}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{1.1.12}$$

$$\tag{1.1.13}$$

$$\text{Spec}^{\text{CS}}\breve{\Phi}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{1.1.14}$$

$$\tag{1.1.15}$$

$$\text{Spec}^{\text{CS}}\Phi_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{1.1.16}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{\psi,\Gamma,A}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{\psi,\Gamma,A}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\text{homotopycolimit}_r M_r, \tag{1.1.17}$$

$$\text{homotopylimit}_I M_I, \tag{1.1.18}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 1.1.4. Similar proposition holds for

$$\text{Quasicoherentsheaves, Perfectcomplex, IndBanach}_*. \tag{1.1.19}$$

Proposition 1.1.5. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Condensed}_* \tag{1.1.20}$$

where $*$ is one of the following spaces:

$$\mathcal{O}_{\text{Spec}}{}^{\text{CS}}\widetilde{\Phi}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{1.1.21}$$

$$\tag{1.1.22}$$

$$\mathcal{O}_{\text{Spec}}{}^{\text{CS}}\breve{\Phi}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{1.1.23}$$

$$\tag{1.1.24}$$

$$\mathcal{O}_{\text{Spec}}{}^{\text{CS}}\Phi_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{1.1.25}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\text{homotopycolimit}_r \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,A}, \text{homotopylimit}_I \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,A},$$

$$\text{homotopycolimit}_r \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\breve{\Phi}^r_{\psi,\Gamma,A}, \text{homotopylimit}_I \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\breve{\Phi}^I_{\psi,\Gamma,A},$$

$$\text{homotopycolimit}_r \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\Phi^r_{\psi,\Gamma,A}, \text{homotopylimit}_I \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\Phi^I_{\psi,\Gamma,A}.$$

$$\text{homotopycolimit}_r \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^r_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \breve{\Phi}^I_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{1.1.26}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{1.1.27}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 1.1.6. Similar proposition holds for

$$\text{Quasicoherentsheaves, IndBanach}_*. \tag{1.1.28}$$

Proposition 1.1.7. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Perfectcomplex, Condensed}_* \tag{1.1.29}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{1.1.30}$$

$$\tag{1.1.31}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{1.1.32}$$

$$\tag{1.1.33}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{1.1.34}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\text{homotopycolimit}}\ \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\breve{\Phi}^{r}_{\psi,\Gamma,A}, \underset{I}{\text{homotopylimit}}\ \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\breve{\Phi}^{I}_{\psi,\Gamma,A},$$

$$\underset{r}{\text{homotopycolimit}}\ \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\Phi^{r}_{\psi,\Gamma,A}, \underset{I}{\text{homotopylimit}}\ \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\Phi^{I}_{\psi,\Gamma,A}.$$

$$\underset{r}{\text{homotopycolimit}}\ \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\widetilde{\Phi}^{r}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\widetilde{\Phi}^{I}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\breve{\Phi}^{r}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \breve{\Phi}^{I}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\Phi^{r}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\Phi^{I}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{1.1.35}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{1.1.36}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 1.1.8. Similar proposition holds for

$$\text{Quasicoherentsheaves, Perfectcomplex, IndBanach}_{*}. \tag{1.1.37}$$

1.1.2 Some Notations

[?] $\mathrm{Spec}^{\mathrm{BK}}$: Bambozzi-Kremnizer ∞-topos.
[?] $\mathrm{Spec}^{\mathrm{CS}}$: Clausen-Scholze ∞-topos.
[?] $\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}$: ∞-ringed structure sheaf of Bambozzi-Kremnizer ∞-topos.
[?] $\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}$: ∞-ringed structure sheaf of Clausen-Scholze ∞-topos.
[?] $\mathrm{Quasicoherentsheaves, IndBanach}_*$: $(\infty,1)$-category of $(\infty,1)$-quasicoherent sheaves of $(\infty,1)$-Banach modules.
[?] $\mathrm{Quasicoherentsheaves, Perfectcomplex, IndBanach}_*$: $(\infty,1)$-category of $(\infty,1)$-quasicoh
erent sheaves of $(\infty,1)$-Banach modules with certain requirement on the perfectness.
[?] $\mathrm{Quasicoherentpresheaves, Condensed}_*$:$(\infty,1)$-category of $(\infty,1)$-quasicoherent sheaves of $(\infty,1)$-condensed solid modules.
[?] $\mathrm{Quasicoherentpresheaves, Perfectcomplex, Condensed}_*$:$(\infty,1)$-category of $(\infty,1)$-quasi
coherent sheaves of $(\infty,1)$-condensed solid modules with certain requirement on the perfectness.
[?] A: Banach Rings.
[?] $-,\circ$: Deformations of Banach Rings as Functors.
[?] X: Preadic Spaces.
[?] $\square$, $X_\square$: Colimits of Rings and Stacks.

1.2 Multivariate Hodge Iwasawa Modules

This chapter follows closely [T1], [T2], [T3], [KPX], [KP], [KL1], [KL2], [BK], [BBBK], [BBM], [KKM], [CS1], [CS2], [CKZ], [PZ], [BCM], [LBV][3].

1.2.1 Frobenius Quasicoherent Modules I

Definition 1.2.1. Let ψ be a toric tower over $\mathbb{Q}_p$ as in [KL2, Chapter 7] with base $\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power:

$$\widetilde{\Delta}_{\psi,\Gamma},\widetilde{\nabla}_{\psi,\Gamma},\widetilde{\Phi}_{\psi,\Gamma},\widetilde{\Delta}^+_{\psi,\Gamma},\widetilde{\nabla}^+_{\psi,\Gamma},\widetilde{\Delta}^\dagger_{\psi,\Gamma},\widetilde{\nabla}^\dagger_{\psi,\Gamma},\widetilde{\Phi}^r_{\psi,\Gamma},\widetilde{\Phi}^I_{\psi,\Gamma},$$

$$\breve{\Delta}_{\psi,\Gamma},\breve{\nabla}_{\psi,\Gamma},\breve{\Phi}_{\psi,\Gamma},\breve{\Delta}^+_{\psi,\Gamma},\breve{\nabla}^+_{\psi,\Gamma},\breve{\Delta}^\dagger_{\psi,\Gamma},\breve{\nabla}^\dagger_{\psi,\Gamma},\breve{\Phi}^r_{\psi,\Gamma},\breve{\Phi}^I_{\psi,\Gamma},$$

$$\Delta_{\psi,\Gamma},\nabla_{\psi,\Gamma},\Phi_{\psi,\Gamma},\Delta^+_{\psi,\Gamma},\nabla^+_{\psi,\Gamma},\Delta^\dagger_{\psi,\Gamma},\nabla^\dagger_{\psi,\Gamma},\Phi^r_{\psi,\Gamma},\Phi^I_{\psi,\Gamma}.$$

[3]Note that all our constructions in this article are motivated by certain deformation needed in our project on the Hodge-Iwasawa theory closely after [BF1], [BF2], [FK], [FS], [He], [KL1], [KL2], [KP], [KPX], [Na1], [Na2], [PR], [RZ], [Sch2], [SW], [Wit]. Also in our context one can study the deformation version of the construction in [LBV] by taking the corresponding de Rham complex of rigid motives over the spaces we consider in this paper.

We now consider the following rings with A being a Banach ring over $\mathbb{Q}_p$. Taking the product we have:

$$\widetilde{\Phi}_{\psi,\Gamma,A}, \widetilde{\Phi}^r_{\psi,\Gamma,A}, \widetilde{\Phi}^I_{\psi,\Gamma,A},$$

$$\breve{\Phi}_{\psi,\Gamma,A}, \breve{\Phi}^r_{\psi,\Gamma,A}, \breve{\Phi}^I_{\psi,\Gamma,A},$$

$$\Phi_{\psi,\Gamma,A}, \Phi^r_{\psi,\Gamma,A}, \Phi^I_{\psi,\Gamma,A}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma := \mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty,1)$-categories of $(\infty,1)$-modules.

Definition 1.2.2. First we consider the Bambozzi-Kremnizer spectrum $\mathrm{Spec}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\Gamma,A}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,A}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,A}, \tag{1.2.1}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,\Gamma,A}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,A}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,A}, \tag{1.2.2}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,\Gamma,A}, \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,A}, \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,A}. \tag{1.2.3}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.4}$$

$$\tag{1.2.5}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.6}$$

$$\tag{1.2.7}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.2.8}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,A}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 1.2.3. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Namely A will still as above as a Banach ring over $\mathbb{Q}_p$. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_{\psi,\Gamma}, \widetilde{\nabla}_{\psi,\Gamma}, \widetilde{\Phi}_{\psi,\Gamma}, \widetilde{\Delta}^+_{\psi,\Gamma}, \widetilde{\nabla}^+_{\psi,\Gamma}, \widetilde{\Delta}^\dagger_{\psi,\Gamma}, \widetilde{\nabla}^\dagger_{\psi,\Gamma}, \widetilde{\Phi}^r_{\psi,\Gamma}, \widetilde{\Phi}^I_{\psi,\Gamma},$$

$$\breve{\Delta}_{\psi,\Gamma}, \breve{\nabla}_{\psi,\Gamma}, \breve{\Phi}_{\psi,\Gamma}, \breve{\Delta}^+_{\psi,\Gamma}, \breve{\nabla}^+_{\psi,\Gamma}, \breve{\Delta}^\dagger_{\psi,\Gamma}, \breve{\nabla}^\dagger_{\psi,\Gamma}, \breve{\Phi}^r_{\psi,\Gamma}, \breve{\Phi}^I_{\psi,\Gamma},$$

$$\Delta_{\psi,\Gamma}, \nabla_{\psi,\Gamma}, \Phi_{\psi,\Gamma}, \Delta^+_{\psi,\Gamma}, \nabla^+_{\psi,\Gamma}, \Delta^\dagger_{\psi,\Gamma}, \nabla^\dagger_{\psi,\Gamma}, \Phi^r_{\psi,\Gamma}, \Phi^I_{\psi,\Gamma},$$

with A. Then we have the notations:

$$\widetilde{\Delta}_{\psi,\Gamma,A}, \widetilde{\nabla}_{\psi,\Gamma,A}, \widetilde{\Phi}_{\psi,\Gamma,A}, \widetilde{\Delta}^+_{\psi,\Gamma,A}, \widetilde{\nabla}^+_{\psi,\Gamma,A}, \widetilde{\Delta}^\dagger_{\psi,\Gamma,A}, \widetilde{\nabla}^\dagger_{\psi,\Gamma,A}, \widetilde{\Phi}^r_{\psi,\Gamma,A}, \widetilde{\Phi}^I_{\psi,\Gamma,A},$$

$$\breve{\Delta}_{\psi,\Gamma,A}, \breve{\nabla}_{\psi,\Gamma,A}, \breve{\Phi}_{\psi,\Gamma,A}, \breve{\Delta}^+_{\psi,\Gamma,A}, \breve{\nabla}^+_{\psi,\Gamma,A}, \breve{\Delta}^\dagger_{\psi,\Gamma,A}, \breve{\nabla}^\dagger_{\psi,\Gamma,A}, \breve{\Phi}^r_{\psi,\Gamma,A}, \breve{\Phi}^I_{\psi,\Gamma,A},$$

$$\Delta_{\psi,\Gamma,A}, \nabla_{\psi,\Gamma,A}, \Phi_{\psi,\Gamma,A}, \Delta^+_{\psi,\Gamma,A}, \nabla^+_{\psi,\Gamma,A}, \Delta^\dagger_{\psi,\Gamma,A}, \nabla^\dagger_{\psi,\Gamma,A}, \Phi^r_{\psi,\Gamma,A}, \Phi^I_{\psi,\Gamma,A}.$$

Definition 1.2.4. First we consider the Clausen-Scholze spectrum $\mathrm{Spec}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,A}, \tag{1.2.9}$$
$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,A}, \tag{1.2.10}$$
$$\tag{1.2.11}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,A}, \breve{\nabla}_{\psi,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,A}, \tag{1.2.12}$$
$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,A}, \breve{\Phi}^I_{\psi,\Gamma,A}, \tag{1.2.13}$$
$$\tag{1.2.14}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,A}, \tag{1.2.15}$$
$$\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,A}. \tag{1.2.16}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.17}$$
$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.18}$$
$$\tag{1.2.19}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.20}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.21}$$

$$\tag{1.2.22}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.23}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.2.24}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,A}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 1.2.5. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\mathrm{Quasicoherentpresheaves}, \mathrm{IndBanach}_* \tag{1.2.25}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.26}$$

$$\tag{1.2.27}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.28}$$

$$\tag{1.2.29}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.2.30}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,A}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,A}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 1.2.6. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves}, \text{Condensed}_* \tag{1.2.31}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.32}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.33}$$

$$\tag{1.2.34}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.35}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.36}$$

$$\tag{1.2.37}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.38}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.2.39}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,A}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,A}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 1.2.7. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Condensed}_* \tag{1.2.40}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.41}$$

$$\tag{1.2.42}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.43}$$

$$\tag{1.2.44}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.45}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,A}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,A}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\, M_r, \tag{1.2.46}$$

$$\underset{I}{\text{homotopylimit}}\, M_I, \tag{1.2.47}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 1.2.8. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{IndBanach}_*. \tag{1.2.48}$$

Definition 1.2.9. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves}, \text{Perfectcomplex}, \text{IndBanach}_* \tag{1.2.49}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{BK}}\widetilde{\Phi}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{1.2.50}$$

$$\tag{1.2.51}$$

$$\text{Spec}^{\text{BK}}\breve{\Phi}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{1.2.52}$$

$$\tag{1.2.53}$$

$$\text{Spec}^{\text{BK}}\Phi_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}. \tag{1.2.54}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\, \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\text{homotopycolimit}}\, \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\text{homotopylimit}}\, \text{Spec}^{\text{BK}}\breve{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\text{homotopycolimit}}\, \text{Spec}^{\text{BK}}\breve{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\text{homotopylimit}}\, \text{Spec}^{\text{BK}}\Phi^r_{\psi,\Gamma,A}, \underset{I}{\text{homotopycolimit}}\, \text{Spec}^{\text{BK}}\Phi^I_{\psi,\Gamma,A}.$$

$$\underset{r}{\text{homotopylimit}}\, \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\, \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\, \text{Spec}^{\text{BK}}\breve{\Phi}^r_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\, \text{Spec}^{\text{BK}}\breve{\Phi}^I_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\, \text{Spec}^{\text{BK}}\Phi^r_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\, \text{Spec}^{\text{BK}}\Phi^I_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}.$$

Definition 1.2.10. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Perfectcomplex, Condensed}_* \tag{1.2.55}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{CS}}\widetilde{\Delta}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Phi}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{1.2.56}$$

$$\text{Spec}^{\text{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{1.2.57}$$

$$\tag{1.2.58}$$

$$\text{Spec}^{\text{CS}}\breve{\Delta}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\breve{\Phi}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\breve{\Delta}^+_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{1.2.59}$$

$$\text{Spec}^{\text{CS}}\breve{\nabla}^+_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{1.2.60}$$

$$\tag{1.2.61}$$

$$\text{Spec}^{\text{CS}}\Delta_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\nabla_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\Phi_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\Delta^+_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{1.2.62}$$

$$\text{Spec}^{\text{CS}}\nabla^+_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\Delta^\dagger_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\nabla^\dagger_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}. \tag{1.2.63}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{\psi,\Gamma,A}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{\psi,\Gamma,A}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}.$$

Proposition 1.2.11. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Perfectcomplex, Condensed}_* \tag{1.2.64}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{CS}}\widetilde{\Phi}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{1.2.65}$$

$$\tag{1.2.66}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.67}$$

$$\tag{1.2.68}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.69}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,A}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{1.2.70}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{1.2.71}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 1.2.12. Similar proposition holds for

$$\mathrm{Quasicoherentsheaves}, \mathrm{Perfectcomplex}, \mathrm{IndBanach}_{*}. \tag{1.2.72}$$

1.2.2 Frobenius Quasicoherent Modules II: Deformation in Banach Rings

Definition 1.2.13. Let ψ be a toric tower over $\mathbb{Q}_p$ as in [KL2, Chapter 7] with base $\mathbb{Q}_p \left\langle X_1^{\pm 1}, ..., X_k^{\pm 1} \right\rangle$. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power:

$$\widetilde{\Delta}_{\psi,\Gamma}, \widetilde{\nabla}_{\psi,\Gamma}, \widetilde{\Phi}_{\psi,\Gamma}, \widetilde{\Delta}^+_{\psi,\Gamma}, \widetilde{\nabla}^+_{\psi,\Gamma}, \widetilde{\Delta}^\dagger_{\psi,\Gamma}, \widetilde{\nabla}^\dagger_{\psi,\Gamma}, \widetilde{\Phi}^r_{\psi,\Gamma}, \widetilde{\Phi}^I_{\psi,\Gamma},$$

$$\breve{\Delta}_{\psi,\Gamma}, \breve{\nabla}_{\psi,\Gamma}, \breve{\Phi}_{\psi,\Gamma}, \breve{\Delta}^+_{\psi,\Gamma}, \breve{\nabla}^+_{\psi,\Gamma}, \breve{\Delta}^\dagger_{\psi,\Gamma}, \breve{\nabla}^\dagger_{\psi,\Gamma}, \breve{\Phi}^r_{\psi,\Gamma}, \breve{\Phi}^I_{\psi,\Gamma},$$

$$\Delta_{\psi,\Gamma}, \nabla_{\psi,\Gamma}, \Phi_{\psi,\Gamma}, \Delta^+_{\psi,\Gamma}, \nabla^+_{\psi,\Gamma}, \Delta^\dagger_{\psi,\Gamma}, \nabla^\dagger_{\psi,\Gamma}, \Phi^r_{\psi,\Gamma}, \Phi^I_{\psi,\Gamma}.$$

We now consider the following rings with $-$ being any deforming Banach ring over $\mathbb{Q}_p$. Taking the product we have:

$$\widetilde{\Phi}_{\psi,\Gamma,-}, \widetilde{\Phi}^r_{\psi,\Gamma,-}, \widetilde{\Phi}^I_{\psi,\Gamma,-},$$

$$\breve{\Phi}_{\psi,\Gamma,-}, \breve{\Phi}^r_{\psi,\Gamma,-}, \breve{\Phi}^I_{\psi,\Gamma,-},$$

$$\Phi_{\psi,\Gamma,-}, \Phi^r_{\psi,\Gamma,-}, \Phi^I_{\psi,\Gamma,-}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma := \mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty, 1)$-categories of $(\infty, 1)$-modules.

Definition 1.2.14. First we consider the Bambozzi-Kremnizer spectrum $\mathrm{Spec}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\Gamma,-}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,-}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,-}, \tag{1.2.73}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,\Gamma,-}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,-}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,-}, \tag{1.2.74}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,\Gamma,-}, \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,-}, \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,-}. \tag{1.2.75}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.76}$$

$$\tag{1.2.77}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.78}$$

$$\tag{1.2.79}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.2.80}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,-}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,-}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 1.2.15. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Namely A will still as above as a Banach ring over $\mathbb{Q}_p$. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_{\psi,\Gamma}, \widetilde{\nabla}_{\psi,\Gamma}, \widetilde{\Phi}_{\psi,\Gamma}, \widetilde{\Delta}^+_{\psi,\Gamma}, \widetilde{\nabla}^+_{\psi,\Gamma}, \widetilde{\Delta}^\dagger_{\psi,\Gamma}, \widetilde{\nabla}^\dagger_{\psi,\Gamma}, \widetilde{\Phi}^r_{\psi,\Gamma}, \widetilde{\Phi}^I_{\psi,\Gamma},$$

$$\breve{\Delta}_{\psi,\Gamma}, \breve{\nabla}_{\psi,\Gamma}, \breve{\Phi}_{\psi,\Gamma}, \breve{\Delta}^+_{\psi,\Gamma}, \breve{\nabla}^+_{\psi,\Gamma}, \breve{\Delta}^\dagger_{\psi,\Gamma}, \breve{\nabla}^\dagger_{\psi,\Gamma}, \breve{\Phi}^r_{\psi,\Gamma}, \breve{\Phi}^I_{\psi,\Gamma},$$

$$\Delta_{\psi,\Gamma}, \nabla_{\psi,\Gamma}, \Phi_{\psi,\Gamma}, \Delta^+_{\psi,\Gamma}, \nabla^+_{\psi,\Gamma}, \Delta^\dagger_{\psi,\Gamma}, \nabla^\dagger_{\psi,\Gamma}, \Phi^r_{\psi,\Gamma}, \Phi^I_{\psi,\Gamma},$$

with A. Then we have the notations:

$$\widetilde{\Delta}_{\psi,\Gamma,-}, \widetilde{\nabla}_{\psi,\Gamma,-}, \widetilde{\Phi}_{\psi,\Gamma,-}, \widetilde{\Delta}^+_{\psi,\Gamma,-}, \widetilde{\nabla}^+_{\psi,\Gamma,-}, \widetilde{\Delta}^\dagger_{\psi,\Gamma,-}, \widetilde{\nabla}^\dagger_{\psi,\Gamma,-}, \widetilde{\Phi}^r_{\psi,\Gamma,-}, \widetilde{\Phi}^I_{\psi,\Gamma,-},$$

$$\breve{\Delta}_{\psi,\Gamma,-}, \breve{\nabla}_{\psi,\Gamma,-}, \breve{\Phi}_{\psi,\Gamma,-}, \breve{\Delta}^+_{\psi,\Gamma,-}, \breve{\nabla}^+_{\psi,\Gamma,-}, \breve{\Delta}^\dagger_{\psi,\Gamma,-}, \breve{\nabla}^\dagger_{\psi,\Gamma,-}, \breve{\Phi}^r_{\psi,\Gamma,-}, \breve{\Phi}^I_{\psi,\Gamma,-},$$

$$\Delta_{\psi,\Gamma,-}, \nabla_{\psi,\Gamma,-}, \Phi_{\psi,\Gamma,-}, \Delta^+_{\psi,\Gamma,-}, \nabla^+_{\psi,\Gamma,-}, \Delta^\dagger_{\psi,\Gamma,-}, \nabla^\dagger_{\psi,\Gamma,-}, \Phi^r_{\psi,\Gamma,-}, \Phi^I_{\psi,\Gamma,-}.$$

Definition 1.2.16. First we consider the Clausen-Scholze spectrum $\mathrm{Spec}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,-}, \tag{1.2.81}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,-}, \tag{1.2.82}$$

$$\tag{1.2.83}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,-}, \breve{\nabla}_{\psi,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,-}, \tag{1.2.84}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,-}, \breve{\Phi}^I_{\psi,\Gamma,-}, \tag{1.2.85}$$

$$\tag{1.2.86}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,-}, \tag{1.2.87}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,-}. \tag{1.2.88}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.89}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.90}$$

$$\tag{1.2.91}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.92}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.93}$$

$$\tag{1.2.94}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.95}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.2.96}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,-}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 1.2.17. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves, IndBanach}_{*} \tag{1.2.97}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{BK}}\widetilde{\Phi}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{1.2.98}$$

$$\tag{1.2.99}$$

$$\text{Spec}^{\text{BK}}\breve{\Phi}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{1.2.100}$$

$$\tag{1.2.101}$$

$$\text{Spec}^{\text{BK}}\Phi_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}. \tag{1.2.102}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\Phi^r_{\psi,\Gamma,-}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\Phi^I_{\psi,\Gamma,-}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^r_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^I_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\Phi^r_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\Phi^I_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}.$$

Definition 1.2.18. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Condensed}_{*} \tag{1.2.103}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{CS}}\widetilde{\Delta}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Phi}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{1.2.104}$$

$$\text{Spec}^{\text{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{1.2.105}$$

$$\tag{1.2.106}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.107}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.108}$$

$$\tag{1.2.109}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.110}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.2.111}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,-}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 1.2.19. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}_{\mathrm{,Condensed}_*} \tag{1.2.112}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.113}$$

$$\tag{1.2.114}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.115}$$

$$\tag{1.2.116}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.117}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we

consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,-}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{1.2.118}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{1.2.119}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 1.2.20. Similar proposition holds for

$$\mathrm{Quasicoherentsheaves}, \mathrm{IndBanach}_*. \tag{1.2.120}$$

Definition 1.2.21. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Perfectcomplex}, \mathrm{IndBanach}_* \tag{1.2.121}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.122}$$

(1.2.123)

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.124}$$
$$\tag{1.2.125}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.2.126}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,-}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 1.2.22. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\mathrm{Quasicoherentsheaves}, \mathrm{Perfectcomplex}, \mathrm{Condensed}_* \tag{1.2.127}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.128}$$
$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.129}$$
$$\tag{1.2.130}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.131}$$
$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.132}$$
$$\tag{1.2.133}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.134}$$
$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.2.135}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{\psi,\Gamma,-}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{\psi,\Gamma,-}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}.$$

Proposition 1.2.23. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Perfectcomplex, Condensed}_* \tag{1.2.136}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{CS}}\widetilde{\Phi}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{1.2.137}$$

$$\tag{1.2.138}$$

$$\text{Spec}^{\text{CS}}\breve{\Phi}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{1.2.139}$$

$$\tag{1.2.140}$$

$$\text{Spec}^{\text{CS}}\Phi_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{1.2.141}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{\psi,\Gamma,-}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{\psi,\Gamma,-}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{1.2.142}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{1.2.143}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 1.2.24. Similar proposition holds for

$$\text{Quasicoherentsheaves, Perfectcomplex, IndBanach}_*. \tag{1.2.144}$$

Then we have the following functoriality results:

Proposition 1.2.25. We have the following commutative diagram:

$$\begin{array}{ccc}
\text{Quasicoherentsheaves, Condensed}_* & \longrightarrow & \text{Quasicoherentsheaves, Condensed}_. \\
\downarrow & & \downarrow \\
\text{Quasicoherentsheaves, Condensed}_{*_{\psi_0}} & \longrightarrow & \text{Quasicoherentsheaves, Condensed}_{\cdot_{\psi_0}}.
\end{array}$$

Proposition 1.2.26. We have the following commutative diagram:

$$\begin{array}{ccc}
\text{Quasicoherentsheaves, IndBanach}_* & \longrightarrow & \text{Quasicoherentsheaves, IndBanach}_. \\
\downarrow & & \downarrow \\
\text{Quasicoherentsheaves, IndBanach}_{*_{\psi_0}} & \longrightarrow & \text{Quasicoherentsheaves, IndBanach}_{\cdot_{\psi_0}}.
\end{array}$$

Proposition 1.2.27. We have the following commutative diagram:

$$
\begin{array}{ccc}
\text{Quasicoherentsheaves, Perfectcomplex, Condensed}_{*} & \longrightarrow & \text{Quasicoherentsheaves, Perfectcomplex, Condensed}_{\cdot} \\
\downarrow & & \downarrow \\
\text{Quasicoherentsheaves, Perfectcomplex, Condensed}_{*\psi_0} & \longrightarrow & \text{Quasicoherentsheaves, Perfectcomplex, Condensed}_{\cdot\psi_0}.
\end{array}
$$

Proposition 1.2.28. We have the following commutative diagram:

$$
\begin{array}{ccc}
\text{Quasicoherentsheaves, Perfectcomplex, IndBanach}_{*} & \longrightarrow & \text{Quasicoherentsheaves, Perfectcomplex, IndBanach}_{\cdot} \\
\downarrow & & \downarrow \\
\text{Quasicoherentsheaves, Perfectcomplex, IndBanach}_{*\psi_0} & \longrightarrow & \text{Quasicoherentsheaves, Perfectcomplex, IndBanach}_{\cdot\psi_0}.
\end{array}
$$

1.2.3 Frobenius Quasicoherent Modules III: Deformation in $(\infty,1)$-Ind-Banach Rings

Definition 1.2.29. Let ψ be a toric tower over $\mathbb{Q}_p$ as in [KL2, Chapter 7] with base $\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power:

$$\widetilde{\Delta}_{\psi,\Gamma},\widetilde{\nabla}_{\psi,\Gamma},\widetilde{\Phi}_{\psi,\Gamma},\widetilde{\Delta}^+_{\psi,\Gamma},\widetilde{\nabla}^+_{\psi,\Gamma},\widetilde{\Delta}^\dagger_{\psi,\Gamma},\widetilde{\nabla}^\dagger_{\psi,\Gamma},\widetilde{\Phi}^r_{\psi,\Gamma},\widetilde{\Phi}^I_{\psi,\Gamma},$$

$$\breve{\Delta}_{\psi,\Gamma},\breve{\nabla}_{\psi,\Gamma},\breve{\Phi}_{\psi,\Gamma},\breve{\Delta}^+_{\psi,\Gamma},\breve{\nabla}^+_{\psi,\Gamma},\breve{\Delta}^\dagger_{\psi,\Gamma},\breve{\nabla}^\dagger_{\psi,\Gamma},\breve{\Phi}^r_{\psi,\Gamma},\breve{\Phi}^I_{\psi,\Gamma},$$

$$\Delta_{\psi,\Gamma},\nabla_{\psi,\Gamma},\Phi_{\psi,\Gamma},\Delta^+_{\psi,\Gamma},\nabla^+_{\psi,\Gamma},\Delta^\dagger_{\psi,\Gamma},\nabla^\dagger_{\psi,\Gamma},\Phi^r_{\psi,\Gamma},\Phi^I_{\psi,\Gamma}.$$

We now consider the following rings with $\square$ being a homotopy colimit

$$\underset{i}{\mathrm{homotopycolimit}}\square_i \tag{1.2.145}$$

of $\mathbb{Q}_p\left\langle Y_1,...,Y_i\right\rangle, i=1,2,...$ in ∞-categories of simplicial ind-Banach rings in [BBBK]

$$\mathrm{SimplicialInd-BanachRings}_{\mathbb{Q}_p} \tag{1.2.146}$$

or animated analytic condensed commutative algebras in [CS2]

$$\mathrm{SimplicialAnalyticCondensed}_{\mathbb{Q}_p}. \tag{1.2.147}$$

Taking the product we have:

$$\widetilde{\Phi}_{\psi,\Gamma,\square},\widetilde{\Phi}^r_{\psi,\Gamma,\square},\widetilde{\Phi}^I_{\psi,\Gamma,\square},$$

$$\breve{\Phi}_{\psi,\Gamma,\square},\breve{\Phi}^r_{\psi,\Gamma,\square},\breve{\Phi}^I_{\psi,\Gamma,\square},$$

$$\Phi_{\psi,\Gamma,\square},\Phi^r_{\psi,\Gamma,\square},\Phi^I_{\psi,\Gamma,\square}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma:=\mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty,1)$-categories of $(\infty,1)$-modules.

Definition 1.2.30. First we consider the Bambozzi-Kremnizer spectrum $\mathrm{Spec}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\Gamma,\square},\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,\square},\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,\square}, \tag{1.2.148}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,\Gamma,\square},\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,\square},\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,\square}, \tag{1.2.149}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,\Gamma,\square},\mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,\square},\mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,\square}. \tag{1.2.150}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.151}$$

$$\tag{1.2.152}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.153}$$

$$\tag{1.2.154}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.2.155}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,\square}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 1.2.31. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Namely A will still as above as a Banach ring over $\mathbb{Q}_p$. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_{\psi,\Gamma}, \widetilde{\nabla}_{\psi,\Gamma}, \widetilde{\Phi}_{\psi,\Gamma}, \widetilde{\Delta}^+_{\psi,\Gamma}, \widetilde{\nabla}^+_{\psi,\Gamma}, \widetilde{\Delta}^\dagger_{\psi,\Gamma}, \widetilde{\nabla}^\dagger_{\psi,\Gamma}, \widetilde{\Phi}^r_{\psi,\Gamma}, \widetilde{\Phi}^I_{\psi,\Gamma},$$

$$\breve{\Delta}_{\psi,\Gamma}, \breve{\nabla}_{\psi,\Gamma}, \breve{\Phi}_{\psi,\Gamma}, \breve{\Delta}^+_{\psi,\Gamma}, \breve{\nabla}^+_{\psi,\Gamma}, \breve{\Delta}^\dagger_{\psi,\Gamma}, \breve{\nabla}^\dagger_{\psi,\Gamma}, \breve{\Phi}^r_{\psi,\Gamma}, \breve{\Phi}^I_{\psi,\Gamma},$$

$$\Delta_{\psi,\Gamma}, \nabla_{\psi,\Gamma}, \Phi_{\psi,\Gamma}, \Delta^+_{\psi,\Gamma}, \nabla^+_{\psi,\Gamma}, \Delta^\dagger_{\psi,\Gamma}, \nabla^\dagger_{\psi,\Gamma}, \Phi^r_{\psi,\Gamma}, \Phi^I_{\psi,\Gamma},$$

with A. Then we have the notations:

$$\widetilde{\Delta}_{\psi,\Gamma,\square}, \widetilde{\nabla}_{\psi,\Gamma,\square}, \widetilde{\Phi}_{\psi,\Gamma,\square}, \widetilde{\Delta}^+_{\psi,\Gamma,\square}, \widetilde{\nabla}^+_{\psi,\Gamma,\square}, \widetilde{\Delta}^\dagger_{\psi,\Gamma,\square}, \widetilde{\nabla}^\dagger_{\psi,\Gamma,\square}, \widetilde{\Phi}^r_{\psi,\Gamma,\square}, \widetilde{\Phi}^I_{\psi,\Gamma,\square},$$

$$\breve{\Delta}_{\psi,\Gamma,\square}, \breve{\nabla}_{\psi,\Gamma,\square}, \breve{\Phi}_{\psi,\Gamma,\square}, \breve{\Delta}^+_{\psi,\Gamma,\square}, \breve{\nabla}^+_{\psi,\Gamma,\square}, \breve{\Delta}^\dagger_{\psi,\Gamma,\square}, \breve{\nabla}^\dagger_{\psi,\Gamma,\square}, \breve{\Phi}^r_{\psi,\Gamma,\square}, \breve{\Phi}^I_{\psi,\Gamma,\square},$$

$$\Delta_{\psi,\Gamma,\square}, \nabla_{\psi,\Gamma,\square}, \Phi_{\psi,\Gamma,\square}, \Delta^+_{\psi,\Gamma,\square}, \nabla^+_{\psi,\Gamma,\square}, \Delta^\dagger_{\psi,\Gamma,\square}, \nabla^\dagger_{\psi,\Gamma,\square}, \Phi^r_{\psi,\Gamma,\square}, \Phi^I_{\psi,\Gamma,\square}.$$

Definition 1.2.32. First we consider the Clausen-Scholze spectrum $\mathrm{Spec}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,\square},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,\square},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,\square},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,\square},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,\square}, \tag{1.2.156}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,\square},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,\square},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\square},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\square}, \tag{1.2.157}$$

$$\tag{1.2.158}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,\square},\breve{\nabla}_{\psi,\Gamma,\square},\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,\square},\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,\square},\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,\square}, \tag{1.2.159}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,\square},\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,\square},\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\square},\breve{\Phi}^I_{\psi,\Gamma,\square}, \tag{1.2.160}$$

$$\tag{1.2.161}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,\square},\mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,\square},\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,\square},\mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,\square},\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,\square}, \tag{1.2.162}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,\square},\mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,\square},\mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\square},\mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\square}. \tag{1.2.163}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.164}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.165}$$

$$\tag{1.2.166}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\breve{\nabla}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.167}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.168}$$

$$\tag{1.2.169}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.170}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.2.171}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r,\bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\,\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\square},\underset{I}{\mathrm{homotopycolimit}}\,\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\,\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\square},\underset{I}{\mathrm{homotopycolimit}}\,\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\,\mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\square},\underset{I}{\mathrm{homotopycolimit}}\,\mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\square}.$$

$$\underset{r}{\mathrm{homotopylimit}}\,\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\underset{I}{\mathrm{homotopycolimit}}\,\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\,\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\underset{I}{\mathrm{homotopycolimit}}\,\breve{\Phi}^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\,\mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\underset{I}{\mathrm{homotopycolimit}}\,\mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 1.2.33. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM][4]:

$$\text{Quasicoherentpresheaves}, \text{IndBanach}_* \tag{1.2.172}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{BK}}\widetilde{\Phi}_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{1.2.173}$$

$$\tag{1.2.174}$$

$$\text{Spec}^{\text{BK}}\breve{\Phi}_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{1.2.175}$$

$$\tag{1.2.176}$$

$$\text{Spec}^{\text{BK}}\Phi_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}. \tag{1.2.177}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^r_{\psi,\Gamma,\square}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\Phi^r_{\psi,\Gamma,\square}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\Phi^I_{\psi,\Gamma,\square}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^r_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^I_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\Phi^r_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\Phi^I_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}.$$

Definition 1.2.34. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves}, \text{Condensed}_* \tag{1.2.178}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{CS}}\widetilde{\Delta}_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Phi}_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{1.2.179}$$

$$\text{Spec}^{\text{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{1.2.180}$$

$$\tag{1.2.181}$$

[4]Here the categories are defined to be the corresponding homotopy colimits of the corresponding categories with respect to each $\square_i$.

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,\square}/\mathrm{Fro}^\mathbb{Z}, \breve{\nabla}_{\psi,\Gamma,\square}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,\square}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,\square}/\mathrm{Fro}^\mathbb{Z}, \tag{1.2.182}$$
$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,\square}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,\square}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,\square}/\mathrm{Fro}^\mathbb{Z}, \tag{1.2.183}$$
$$\tag{1.2.184}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,\square}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,\square}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,\square}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,\square}/\mathrm{Fro}^\mathbb{Z}, \tag{1.2.185}$$
$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,\square}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,\square}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,\square}/\mathrm{Fro}^\mathbb{Z}. \tag{1.2.186}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\square}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}/\mathrm{Fro}^\mathbb{Z}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\square}/\mathrm{Fro}^\mathbb{Z},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\square}/\mathrm{Fro}^\mathbb{Z}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,\square}/\mathrm{Fro}^\mathbb{Z},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\square}/\mathrm{Fro}^\mathbb{Z}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\square}/\mathrm{Fro}^\mathbb{Z}.$$

Proposition 1.2.35. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Condensed}_* \tag{1.2.187}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,\square}/\mathrm{Fro}^\mathbb{Z}, \tag{1.2.188}$$
$$\tag{1.2.189}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,\square}/\mathrm{Fro}^\mathbb{Z}, \tag{1.2.190}$$
$$\tag{1.2.191}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,\square}/\mathrm{Fro}^\mathbb{Z}, \tag{1.2.192}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we

consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\square}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{1.2.193}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{1.2.194}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 1.2.36. Similar proposition holds for

$$\mathrm{Quasicoherentsheaves}, \mathrm{IndBanach}_*. \tag{1.2.195}$$

Definition 1.2.37. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Perfectcomplex}, \mathrm{IndBanach}_* \tag{1.2.196}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.197}$$

(1.2.198)

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.199}$$

$$\tag{1.2.200}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.2.201}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,\square}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 1.2.38. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\mathrm{Quasicoherentsheaves}, \mathrm{Perfectcomplex}, \mathrm{Condensed}_* \tag{1.2.202}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.203}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.204}$$

$$\tag{1.2.205}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.206}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.207}$$

$$\tag{1.2.208}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.209}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.2.210}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\square}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 1.2.39. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Perfectcomplex}, \mathrm{Condensed}_* \tag{1.2.211}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.212}$$

(1.2.213)

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.214}$$

(1.2.215)

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.2.216}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\square}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{1.2.217}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{1.2.218}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 1.2.40. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{Perfectcomplex}, \text{IndBanach}_*. \tag{1.2.219}$$

1.3 Univariate Hodge Iwasawa Modules

This chapter follows closely [T1], [T2], [T3], [KPX], [KP], [KL1], [KL2], [BK], [BBBK], [BBM], [KKM], [CS1], [CS2], [LBV].

1.3.1 Frobenius Quasicoherent Modules I

Definition 1.3.1. Let ψ be a toric tower over $\mathbb{Q}_p$ as in [KL2, Chapter 7] with base $\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power[5]:

$$\widetilde{\Delta}_\psi, \widetilde{\nabla}_\psi, \widetilde{\Phi}_\psi, \widetilde{\Delta}^+_\psi, \widetilde{\nabla}^+_\psi, \widetilde{\Delta}^\dagger_\psi, \widetilde{\nabla}^\dagger_\psi, \widetilde{\Phi}^r_\psi, \widetilde{\Phi}^I_\psi,$$

$$\breve{\Delta}_\psi, \breve{\nabla}_\psi, \breve{\Phi}_\psi, \breve{\Delta}^+_\psi, \breve{\nabla}^+_\psi, \breve{\Delta}^\dagger_\psi, \breve{\nabla}^\dagger_\psi, \breve{\Phi}^r_\psi, \breve{\Phi}^I_\psi,$$

$$\Delta_\psi, \nabla_\psi, \Phi_\psi, \Delta^+_\psi, \nabla^+_\psi, \Delta^\dagger_\psi, \nabla^\dagger_\psi, \Phi^r_\psi, \Phi^I_\psi.$$

We now consider the following rings with A being a Banach ring over $\mathbb{Q}_p$. Taking the product we have:

$$\widetilde{\Phi}_{\psi,A}, \widetilde{\Phi}^r_{\psi,A}, \widetilde{\Phi}^I_{\psi,A},$$

[5] Here $|\Gamma| = 1$.

$$\breve{\Phi}_{\psi,A}, \breve{\Phi}^r_{\psi,A}, \breve{\Phi}^I_{\psi,A},$$

$$\Phi_{\psi,A}, \Phi^r_{\psi,A}, \Phi^I_{\psi,A}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma := \mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty,1)$-categories of $(\infty,1)$-modules.

Definition 1.3.2. First we consider the Bambozzi-Kremnizer spectrum $\mathrm{Spec}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,A}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,A}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,A}, \tag{1.3.1}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,A}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,A}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,A}, \tag{1.3.2}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,A}, \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,A}, \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,A}. \tag{1.3.3}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.4}$$

(1.3.5)

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.6}$$

(1.3.7)

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.3.8}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,A}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 1.3.3. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Namely A will still as above as a Banach ring over $\mathbb{Q}_p$. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_\psi, \widetilde{\nabla}_\psi, \widetilde{\Phi}_\psi, \widetilde{\Delta}^+_\psi, \widetilde{\nabla}^+_\psi, \widetilde{\Delta}^\dagger_\psi, \widetilde{\nabla}^\dagger_\psi, \widetilde{\Phi}^r_\psi, \widetilde{\Phi}^I_\psi,$$

$$\breve{\Delta}_\psi, \breve{\nabla}_\psi, \breve{\Phi}_\psi, \breve{\Delta}^+_\psi, \breve{\nabla}^+_\psi, \breve{\Delta}^\dagger_\psi, \breve{\nabla}^\dagger_\psi, \breve{\Phi}^r_\psi, \breve{\Phi}^I_\psi,$$

$$\Delta_\psi, \nabla_\psi, \Phi_\psi, \Delta^+_\psi, \nabla^+_\psi, \Delta^\dagger_\psi, \nabla^\dagger_\psi, \Phi^r_\psi, \Phi^I_\psi,$$

with A. Then we have the notations:

$$\widetilde{\Delta}_{\psi,A}, \widetilde{\nabla}_{\psi,A}, \widetilde{\Phi}_{\psi,A}, \widetilde{\Delta}^+_{\psi,A}, \widetilde{\nabla}^+_{\psi,A}, \widetilde{\Delta}^\dagger_{\psi,A}, \widetilde{\nabla}^\dagger_{\psi,A}, \widetilde{\Phi}^r_{\psi,A}, \widetilde{\Phi}^I_{\psi,A},$$

$$\breve{\Delta}_{\psi,A}, \breve{\nabla}_{\psi,A}, \breve{\Phi}_{\psi,A}, \breve{\Delta}^+_{\psi,A}, \breve{\nabla}^+_{\psi,A}, \breve{\Delta}^\dagger_{\psi,A}, \breve{\nabla}^\dagger_{\psi,A}, \breve{\Phi}^r_{\psi,A}, \breve{\Phi}^I_{\psi,A},$$

$$\Delta_{\psi,A}, \nabla_{\psi,A}, \Phi_{\psi,A}, \Delta^+_{\psi,A}, \nabla^+_{\psi,A}, \Delta^\dagger_{\psi,A}, \nabla^\dagger_{\psi,A}, \Phi^r_{\psi,A}, \Phi^I_{\psi,A}.$$

Definition 1.3.4. First we consider the Clausen-Scholze spectrum $\mathrm{Spec}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,A}, \tag{1.3.9}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,A}, \tag{1.3.10}$$

$$\tag{1.3.11}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,A}, \breve{\nabla}_{\psi,A}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,A}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,A}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,A}, \tag{1.3.12}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,A}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,A}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,A}, \breve{\Phi}^I_{\psi,A}, \tag{1.3.13}$$

$$\tag{1.3.14}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,A}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,A}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,A}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,A}, \mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,A}, \tag{1.3.15}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,A}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,A}, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,A}, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,A}. \tag{1.3.16}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.17}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.18}$$

$$\tag{1.3.19}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.20}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.21}$$

$$\tag{1.3.22}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.23}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.3.24}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,A}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 1.3.5. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\mathrm{Quasicoherentpresheaves}, \mathrm{IndBanach}_{*} \tag{1.3.25}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.26}$$

$$\tag{1.3.27}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.28}$$

$$\tag{1.3.29}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.3.30}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{\psi,A}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{\psi,A},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^r_{\psi,A}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^I_{\psi,A},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\Phi^r_{\psi,A}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\Phi^I_{\psi,A}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{\psi,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^r_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^I_{\psi,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\Phi^r_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\Phi^I_{\psi,A}/\text{Fro}^{\mathbb{Z}}.$$

Definition 1.3.6. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Condensed}_* \tag{1.3.31}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{CS}}\widetilde{\Delta}_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Phi}_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^+_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \tag{1.3.32}$$

$$\text{Spec}^{\text{CS}}\widetilde{\nabla}^+_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^\dagger_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}^\dagger_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \tag{1.3.33}$$

$$\tag{1.3.34}$$

$$\text{Spec}^{\text{CS}}\breve{\Delta}_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\breve{\Phi}_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\breve{\Delta}^+_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \tag{1.3.35}$$

$$\text{Spec}^{\text{CS}}\breve{\nabla}^+_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\breve{\Delta}^\dagger_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\breve{\nabla}^\dagger_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \tag{1.3.36}$$

$$\tag{1.3.37}$$

$$\text{Spec}^{\text{CS}}\Delta_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\nabla_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\Phi_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\Delta^+_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \tag{1.3.38}$$

$$\text{Spec}^{\text{CS}}\nabla^+_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\Delta^\dagger_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\nabla^\dagger_{\psi,A}/\text{Fro}^{\mathbb{Z}}. \tag{1.3.39}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{\psi,A}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{\psi,A},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{\psi,A}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^I_{\psi,A},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,A}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,A}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 1.3.7. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Condensed}_* \tag{1.3.40}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.41}$$

$$\tag{1.3.42}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.43}$$

$$\tag{1.3.44}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.45}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,A}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,A},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,A}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,A},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,A}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,A}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\, M_r, \tag{1.3.46}$$

$$\underset{I}{\mathrm{homotopylimit}}\, M_I, \tag{1.3.47}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 1.3.8. Similar proposition holds for

$$\mathrm{Quasicoherentsheaves}, \mathrm{IndBanach}_*. \tag{1.3.48}$$

Definition 1.3.9. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Perfectcomplex}, \mathrm{IndBanach}_* \tag{1.3.49}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.50}$$

$$\tag{1.3.51}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.52}$$

$$\tag{1.3.53}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.3.54}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,A}, \underset{I}{\mathrm{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,A}, \underset{I}{\mathrm{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\, \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,A}, \underset{I}{\mathrm{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,A}.$$

$$\underset{r}{\mathrm{homotopylimit}}\, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\, \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 1.3.10. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Perfectcomplex, Condensed}_* \tag{1.3.55}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.56}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.57}$$

$$\tag{1.3.58}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.59}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.60}$$

$$\tag{1.3.61}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.62}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.3.63}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,A}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,A},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,A}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,A},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,A}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,A}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 1.3.11. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Perfectcomplex, Condensed}_* \tag{1.3.64}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.65}$$

$$\tag{1.3.66}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.67}$$

$$\tag{1.3.68}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.69}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,A}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{1.3.70}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{1.3.71}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 1.3.12. Similar proposition holds for

$$\mathrm{Quasicoherentsheaves}, \mathrm{Perfectcomplex}, \mathrm{IndBanach}_*. \tag{1.3.72}$$

1.3.2 Frobenius Quasicoherent Modules II: Deformation in Banach Rings

Definition 1.3.13. Let ψ be a toric tower over $\mathbb{Q}_p$ as in [KL2, Chapter 7] with base $\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power[6]:

$$\widetilde{\Delta}_\psi, \widetilde{\nabla}_\psi, \widetilde{\Phi}_\psi, \widetilde{\Delta}^+_\psi, \widetilde{\nabla}^+_\psi, \widetilde{\Delta}^\dagger_\psi, \widetilde{\nabla}^\dagger_\psi, \widetilde{\Phi}^r_\psi, \widetilde{\Phi}^I_\psi,$$

$$\breve{\Delta}_\psi, \breve{\nabla}_\psi, \breve{\Phi}_\psi, \breve{\Delta}^+_\psi, \breve{\nabla}^+_\psi, \breve{\Delta}^\dagger_\psi, \breve{\nabla}^\dagger_\psi, \breve{\Phi}^r_\psi, \breve{\Phi}^I_\psi,$$

$$\Delta_\psi, \nabla_\psi, \Phi_\psi, \Delta^+_\psi, \nabla^+_\psi, \Delta^\dagger_\psi, \nabla^\dagger_\psi, \Phi^r_\psi, \Phi^I_\psi.$$

We now consider the following rings with $-$ being any deforming Banach ring over $\mathbb{Q}_p$. Taking the product we have:

$$\widetilde{\Phi}_{\psi,-}, \widetilde{\Phi}^r_{\psi,-}, \widetilde{\Phi}^I_{\psi,-},$$

$$\breve{\Phi}_{\psi,-}, \breve{\Phi}^r_{\psi,-}, \breve{\Phi}^I_{\psi,-},$$

$$\Phi_{\psi,-}, \Phi^r_{\psi,-}, \Phi^I_{\psi,-}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma := \mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty,1)$-categories of $(\infty,1)$-modules.

Definition 1.3.14. First we consider the Bambozzi-Kremnizer spectrum $\mathrm{Spec}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,-}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,-}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,-}, \tag{1.3.73}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,-}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,-}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,-}, \tag{1.3.74}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,-}, \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,-}, \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,-}. \tag{1.3.75}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.76}$$

$$\tag{1.3.77}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.78}$$

$$\tag{1.3.79}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.3.80}$$

[6] Here $|\Gamma| = 1$.

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,-}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,-},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,-}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,-},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,-}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,-}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 1.3.15. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Namely A will still as above as a Banach ring over $\mathbb{Q}_p$. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_\psi, \widetilde{\nabla}_\psi, \widetilde{\Phi}_\psi, \widetilde{\Delta}^+_\psi, \widetilde{\nabla}^+_\psi, \widetilde{\Delta}^\dagger_\psi, \widetilde{\nabla}^\dagger_\psi, \widetilde{\Phi}^r_\psi, \widetilde{\Phi}^I_\psi,$$

$$\breve{\Delta}_\psi, \breve{\nabla}_\psi, \breve{\Phi}_\psi, \breve{\Delta}^+_\psi, \breve{\nabla}^+_\psi, \breve{\Delta}^\dagger_\psi, \breve{\nabla}^\dagger_\psi, \breve{\Phi}^r_\psi, \breve{\Phi}^I_\psi,$$

$$\Delta_\psi, \nabla_\psi, \Phi_\psi, \Delta^+_\psi, \nabla^+_\psi, \Delta^\dagger_\psi, \nabla^\dagger_\psi, \Phi^r_\psi, \Phi^I_\psi,$$

with A. Then we have the notations:

$$\widetilde{\Delta}_{\psi,-}, \widetilde{\nabla}_{\psi,-}, \widetilde{\Phi}_{\psi,-}, \widetilde{\Delta}^+_{\psi,-}, \widetilde{\nabla}^+_{\psi,-}, \widetilde{\Delta}^\dagger_{\psi,-}, \widetilde{\nabla}^\dagger_{\psi,-}, \widetilde{\Phi}^r_{\psi,-}, \widetilde{\Phi}^I_{\psi,-},$$

$$\breve{\Delta}_{\psi,-}, \breve{\nabla}_{\psi,-}, \breve{\Phi}_{\psi,-}, \breve{\Delta}^+_{\psi,-}, \breve{\nabla}^+_{\psi,-}, \breve{\Delta}^\dagger_{\psi,-}, \breve{\nabla}^\dagger_{\psi,-}, \breve{\Phi}^r_{\psi,-}, \breve{\Phi}^I_{\psi,-},$$

$$\Delta_{\psi,-}, \nabla_{\psi,-}, \Phi_{\psi,-}, \Delta^+_{\psi,-}, \nabla^+_{\psi,-}, \Delta^\dagger_{\psi,-}, \nabla^\dagger_{\psi,-}, \Phi^r_{\psi,-}, \Phi^I_{\psi,-}.$$

Definition 1.3.16. First we consider the Clausen-Scholze spectrum $\mathrm{Spec}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,-}, \tag{1.3.81}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,-}, \tag{1.3.82}$$

$$\tag{1.3.83}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,-}, \breve{\nabla}_{\psi,-}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,-}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,-}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,-}, \tag{1.3.84}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,-}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,-}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,-}, \breve{\Phi}^I_{\psi,-}, \tag{1.3.85}$$

$$\tag{1.3.86}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,-}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,-}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,-}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,-}, \mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,-}, \tag{1.3.87}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,-}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,-}, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,-}, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,-}. \tag{1.3.88}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.89}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.90}$$

$$\tag{1.3.91}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.92}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.93}$$

$$\tag{1.3.94}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.95}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.3.96}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,-},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,-},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,-}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 1.3.17. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves, IndBanach}_* \tag{1.3.97}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{BK}}\widetilde{\Phi}_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \tag{1.3.98}$$

$$\tag{1.3.99}$$

$$\text{Spec}^{\text{BK}}\breve{\Phi}_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \tag{1.3.100}$$

$$\tag{1.3.101}$$

$$\text{Spec}^{\text{BK}}\Phi_{\psi,-}/\text{Fro}^{\mathbb{Z}}. \tag{1.3.102}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{\psi,-}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{\psi,-},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^r_{\psi,-}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^I_{\psi,-},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\Phi^r_{\psi,-}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\Phi^I_{\psi,-}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{\psi,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^r_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^I_{\psi,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\Phi^r_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\Phi^I_{\psi,-}/\text{Fro}^{\mathbb{Z}}.$$

Definition 1.3.18. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Condensed}_* \tag{1.3.103}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{CS}}\widetilde{\Delta}_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Phi}_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^+_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \tag{1.3.104}$$

$$\text{Spec}^{\text{CS}}\widetilde{\nabla}^+_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^\dagger_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}^\dagger_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \tag{1.3.105}$$

$$\tag{1.3.106}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},\breve{\nabla}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^{+}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \quad (1.3.107)$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^{+}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^{\dagger}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^{\dagger}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \quad (1.3.108)$$

$$(1.3.109)$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\Delta^{+}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \quad (1.3.110)$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^{+}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\Delta^{\dagger}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\nabla^{\dagger}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}. \quad (1.3.111)$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{\psi,-},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{\psi,-},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{r}_{\psi,-},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{I}_{\psi,-},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{r}_{\psi,-},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{I}_{\psi,-}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{r}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},\underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^{I}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{r}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{I}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 1.3.19. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}_{\mathrm{,Condensed}_*} \quad (1.3.112)$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \quad (1.3.113)$$

$$(1.3.114)$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \quad (1.3.115)$$

$$(1.3.116)$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \quad (1.3.117)$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we

consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{\psi,-}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{\psi,-},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{\psi,-}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^I_{\psi,-},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{\psi,-}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{\psi,-}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{\psi,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{\psi,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{\psi,-}/\text{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{1.3.118}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{1.3.119}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 1.3.20. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{IndBanach}_*. \tag{1.3.120}$$

Definition 1.3.21. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves}, \text{Perfectcomplex}, \text{IndBanach}_* \tag{1.3.121}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{BK}}\widetilde{\Phi}_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \tag{1.3.122}$$

$$\tag{1.3.123}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.124}$$
$$\tag{1.3.125}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.3.126}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,-},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,-},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,-}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 1.3.22. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\mathrm{Quasicoherentsheaves}, \mathrm{Perfectcomplex}, \mathrm{Condensed}_* \tag{1.3.127}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.128}$$
$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.129}$$
$$\tag{1.3.130}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.131}$$
$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.132}$$
$$\tag{1.3.133}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.134}$$
$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.3.135}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,-}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,-},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,-}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,-},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,-}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,-}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 1.3.23. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Perfectcomplex, Condensed}_* \tag{1.3.136}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.137}$$

$$\tag{1.3.138}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.139}$$

$$\tag{1.3.140}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.3.141}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,-}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,-},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,-}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,-},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,-}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,-}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{1.3.142}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{1.3.143}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 1.3.24. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{Perfectcomplex}, \text{IndBanach}_{*}. \tag{1.3.144}$$

1.4 Multivariate Hodge Iwasawa Prestacks

This chapter follows closely [T1], [T2], [T3], [KPX], [KP], [KL1], [KL2], [BK], [BBBK], [BBM], [KKM], [CS1], [CS2], [CKZ], [PZ], [BCM], [LBV].

1.4.1 Frobenius Quasicoherent Prestacks I

Definition 1.4.1. We now consider the pro-étale site of $\mathrm{Spa}\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$ from [Sch], denote that by $*$. To be more accurate we replace one component for Γ with the pro-étale site of $\mathrm{Spa}\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$. And we treat then all the functor to be prestacks for this site[7]. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power:

$$\widetilde{\Delta}_{*,\Gamma}, \widetilde{\nabla}_{*,\Gamma}, \widetilde{\Phi}_{*,\Gamma}, \widetilde{\Delta}^+_{*,\Gamma}, \widetilde{\nabla}^+_{*,\Gamma}, \widetilde{\Delta}^\dagger_{*,\Gamma}, \widetilde{\nabla}^\dagger_{*,\Gamma}, \widetilde{\Phi}^r_{*,\Gamma}, \widetilde{\Phi}^I_{*,\Gamma},$$

$$\breve{\Delta}_{*,\Gamma}, \breve{\nabla}_{*,\Gamma}, \breve{\Phi}_{*,\Gamma}, \breve{\Delta}^+_{*,\Gamma}, \breve{\nabla}^+_{*,\Gamma}, \breve{\Delta}^\dagger_{*,\Gamma}, \breve{\nabla}^\dagger_{*,\Gamma}, \breve{\Phi}^r_{*,\Gamma}, \breve{\Phi}^I_{*,\Gamma},$$

$$\Delta_{*,\Gamma}, \nabla_{*,\Gamma}, \Phi_{*,\Gamma}, \Delta^+_{*,\Gamma}, \nabla^+_{*,\Gamma}, \Delta^\dagger_{*,\Gamma}, \nabla^\dagger_{*,\Gamma}, \Phi^r_{*,\Gamma}, \Phi^I_{*,\Gamma}.$$

[7]Here for those imperfect rings, the notation will mean that the specific component forming the pro-étale site will be the perfect version of the corresponding ring. Certainly if we have $|\Gamma| = 1$ then we have that all the rings are perfect in [KL1] and [KL2].

We now consider the following rings with A being a Banach ring over $\mathbb{Q}_p$. Taking the product we have:

$$\widetilde{\Phi}_{*,\Gamma,A}, \widetilde{\Phi}^r_{*,\Gamma,A}, \widetilde{\Phi}^I_{*,\Gamma,A},$$

$$\breve{\Phi}_{*,\Gamma,A}, \breve{\Phi}^r_{*,\Gamma,A}, \breve{\Phi}^I_{*,\Gamma,A},$$

$$\Phi_{*,\Gamma,A}, \Phi^r_{*,\Gamma,A}, \Phi^I_{*,\Gamma,A}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma := \mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty,1)$-categories of $(\infty,1)$-modules.

Definition 1.4.2. First we consider the Bambozzi-Kremnizer spectrum $\mathrm{Spec}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,\Gamma,A}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,A}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,A}, \tag{1.4.1}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,\Gamma,A}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,A}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,A}, \tag{1.4.2}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,\Gamma,A}, \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,A}, \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,A}. \tag{1.4.3}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.4}$$

$$\tag{1.4.5}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.6}$$

$$\tag{1.4.7}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.4.8}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,A}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 1.4.3. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Namely A will still as above as a Banach ring over $\mathbb{Q}_p$. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_{*,\Gamma}, \widetilde{\nabla}_{*,\Gamma}, \widetilde{\Phi}_{*,\Gamma}, \widetilde{\Delta}^+_{*,\Gamma}, \widetilde{\nabla}^+_{*,\Gamma}, \widetilde{\Delta}^\dagger_{*,\Gamma}, \widetilde{\nabla}^\dagger_{*,\Gamma}, \widetilde{\Phi}^r_{*,\Gamma}, \widetilde{\Phi}^I_{*,\Gamma},$$

$$\breve{\Delta}_{*,\Gamma}, \breve{\nabla}_{*,\Gamma}, \breve{\Phi}_{*,\Gamma}, \breve{\Delta}^+_{*,\Gamma}, \breve{\nabla}^+_{*,\Gamma}, \breve{\Delta}^\dagger_{*,\Gamma}, \breve{\nabla}^\dagger_{*,\Gamma}, \breve{\Phi}^r_{*,\Gamma}, \breve{\Phi}^I_{*,\Gamma},$$

$$\Delta_{*,\Gamma}, \nabla_{*,\Gamma}, \Phi_{*,\Gamma}, \Delta^+_{*,\Gamma}, \nabla^+_{*,\Gamma}, \Delta^\dagger_{*,\Gamma}, \nabla^\dagger_{*,\Gamma}, \Phi^r_{*,\Gamma}, \Phi^I_{*,\Gamma},$$

with A. Then we have the notations:

$$\widetilde{\Delta}_{*,\Gamma,A}, \widetilde{\nabla}_{*,\Gamma,A}, \widetilde{\Phi}_{*,\Gamma,A}, \widetilde{\Delta}^+_{*,\Gamma,A}, \widetilde{\nabla}^+_{*,\Gamma,A}, \widetilde{\Delta}^\dagger_{*,\Gamma,A}, \widetilde{\nabla}^\dagger_{*,\Gamma,A}, \widetilde{\Phi}^r_{*,\Gamma,A}, \widetilde{\Phi}^I_{*,\Gamma,A},$$

$$\breve{\Delta}_{*,\Gamma,A}, \breve{\nabla}_{*,\Gamma,A}, \breve{\Phi}_{*,\Gamma,A}, \breve{\Delta}^+_{*,\Gamma,A}, \breve{\nabla}^+_{*,\Gamma,A}, \breve{\Delta}^\dagger_{*,\Gamma,A}, \breve{\nabla}^\dagger_{*,\Gamma,A}, \breve{\Phi}^r_{*,\Gamma,A}, \breve{\Phi}^I_{*,\Gamma,A},$$

$$\Delta_{*,\Gamma,A}, \nabla_{*,\Gamma,A}, \Phi_{*,\Gamma,A}, \Delta^+_{*,\Gamma,A}, \nabla^+_{*,\Gamma,A}, \Delta^\dagger_{*,\Gamma,A}, \nabla^\dagger_{*,\Gamma,A}, \Phi^r_{*,\Gamma,A}, \Phi^I_{*,\Gamma,A}.$$

Definition 1.4.4. First we consider the Clausen-Scholze spectrum $\mathrm{Spec}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,\Gamma,A}, \tag{1.4.9}$$
$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,A}, \tag{1.4.10}$$
$$\tag{1.4.11}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,A}, \breve{\nabla}_{*,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,\Gamma,A}, \tag{1.4.12}$$
$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,A}, \breve{\Phi}^I_{*,\Gamma,A}, \tag{1.4.13}$$
$$\tag{1.4.14}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,\Gamma,A}, \tag{1.4.15}$$
$$\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,A}, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,A}. \tag{1.4.16}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.17}$$
$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.18}$$
$$\tag{1.4.19}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,A}/\mathrm{Fro}^\mathbb{Z}, \breve{\nabla}_{*,\Gamma,A}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,A}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,\Gamma,A}/\mathrm{Fro}^\mathbb{Z}, \tag{1.4.20}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,\Gamma,A}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,\Gamma,A}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,\Gamma,A}/\mathrm{Fro}^\mathbb{Z}, \tag{1.4.21}$$

$$\tag{1.4.22}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\Gamma,A}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\Gamma,A}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,A}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,\Gamma,A}/\mathrm{Fro}^\mathbb{Z}, \tag{1.4.23}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,\Gamma,A}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,\Gamma,A}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,\Gamma,A}/\mathrm{Fro}^\mathbb{Z}. \tag{1.4.24}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,A}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,A}/\mathrm{Fro}^\mathbb{Z}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,A}/\mathrm{Fro}^\mathbb{Z},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,A}/\mathrm{Fro}^\mathbb{Z}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{*,\Gamma,A}/\mathrm{Fro}^\mathbb{Z},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,A}/\mathrm{Fro}^\mathbb{Z}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,A}/\mathrm{Fro}^\mathbb{Z}.$$

Definition 1.4.5. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\mathrm{Quasicoherentpresheaves}, \mathrm{IndBanach}_* \tag{1.4.25}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,\Gamma,A}/\mathrm{Fro}^\mathbb{Z}, \tag{1.4.26}$$

$$\tag{1.4.27}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,\Gamma,A}/\mathrm{Fro}^\mathbb{Z}, \tag{1.4.28}$$

$$\tag{1.4.29}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,\Gamma,A}/\mathrm{Fro}^\mathbb{Z}. \tag{1.4.30}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,A}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 1.4.6. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\mathrm{Quasicoherentsheaves,Condensed}_{*} \tag{1.4.31}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.32}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.33}$$

$$\tag{1.4.34}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.35}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.36}$$

$$\tag{1.4.37}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.38}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.4.39}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,A}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 1.4.7. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Condensed}_{*} \tag{1.4.40}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.41}$$

$$\tag{1.4.42}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.43}$$

$$\tag{1.4.44}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.45}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{I}_{*,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{r}_{*,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{I}_{*,\Gamma,A}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^{I}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{r}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{I}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{1.4.46}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{1.4.47}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 1.4.8. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{IndBanach}_*. \tag{1.4.48}$$

Definition 1.4.9. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves}, \text{Perfectcomplex}, \text{IndBanach}_* \tag{1.4.49}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{BK}}\widetilde{\Phi}_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{1.4.50}$$

$$\tag{1.4.51}$$

$$\text{Spec}^{\text{BK}}\breve{\Phi}_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{1.4.52}$$

$$\tag{1.4.53}$$

$$\text{Spec}^{\text{BK}}\Phi_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}. \tag{1.4.54}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{*,\Gamma,A}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{*,\Gamma,A},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^r_{*,\Gamma,A}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^I_{*,\Gamma,A},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\Phi^r_{*,\Gamma,A}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\Phi^I_{*,\Gamma,A}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^r_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^I_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\Phi^r_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\Phi^I_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}.$$

Definition 1.4.10. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves}, \text{Perfectcomplex}, \text{Condensed}_* \tag{1.4.55}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^{+}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.56}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^{+}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^{\dagger}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^{\dagger}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.57}$$

$$\tag{1.4.58}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^{+}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.59}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^{+}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^{\dagger}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^{\dagger}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.60}$$

$$\tag{1.4.61}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^{+}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.62}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^{+}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^{\dagger}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^{\dagger}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.4.63}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{I}_{*,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{r}_{*,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{I}_{*,\Gamma,A}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^{I}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{r}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{I}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 1.4.11. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves, Perfectcomplex, Condensed}_{*} \tag{1.4.64}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.65}$$

$$\tag{1.4.66}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.67}$$

$$\tag{1.4.68}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.69}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,A}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{1.4.70}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{1.4.71}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 1.4.12. Similar proposition holds for

$$\mathrm{Quasicoherentsheaves}, \mathrm{Perfectcomplex}, \mathrm{IndBanach}_*. \tag{1.4.72}$$

1.4.2 Frobenius Quasicoherent Prestacks II: Deformation in Banach Rings

Definition 1.4.13. We now consider the pro-étale site of $\mathrm{Spa}\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$, denote that by $*$. To be more accurate we replace one component for Γ with the pro-étale site of $\mathrm{Spa}\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$. And we treat then all the functor to be prestacks for this site. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power:

$$\widetilde{\Delta}_{*,\Gamma},\widetilde{\nabla}_{*,\Gamma},\widetilde{\Phi}_{*,\Gamma},\widetilde{\Delta}^+_{*,\Gamma},\widetilde{\nabla}^+_{*,\Gamma},\widetilde{\Delta}^\dagger_{*,\Gamma},\widetilde{\nabla}^\dagger_{*,\Gamma},\widetilde{\Phi}^r_{*,\Gamma},\widetilde{\Phi}^I_{*,\Gamma},$$

$$\breve{\Delta}_{*,\Gamma},\breve{\nabla}_{*,\Gamma},\breve{\Phi}_{*,\Gamma},\breve{\Delta}^+_{*,\Gamma},\breve{\nabla}^+_{*,\Gamma},\breve{\Delta}^\dagger_{*,\Gamma},\breve{\nabla}^\dagger_{*,\Gamma},\breve{\Phi}^r_{*,\Gamma},\breve{\Phi}^I_{*,\Gamma},$$

$$\Delta_{*,\Gamma},\nabla_{*,\Gamma},\Phi_{*,\Gamma},\Delta^+_{*,\Gamma},\nabla^+_{*,\Gamma},\Delta^\dagger_{*,\Gamma},\nabla^\dagger_{*,\Gamma},\Phi^r_{*,\Gamma},\Phi^I_{*,\Gamma}.$$

We now consider the following rings with $-$ being any deforming Banach ring over $\mathbb{Q}_p$. Taking the product we have:

$$\widetilde{\Phi}_{*,\Gamma,-},\widetilde{\Phi}^r_{*,\Gamma,-},\widetilde{\Phi}^I_{*,\Gamma,-},$$

$$\breve{\Phi}_{*,\Gamma,-},\breve{\Phi}^r_{*,\Gamma,-},\breve{\Phi}^I_{*,\Gamma,-},$$

$$\Phi_{*,\Gamma,-},\Phi^r_{*,\Gamma,-},\Phi^I_{*,\Gamma,-}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma := \mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty,1)$-categories of $(\infty,1)$-modules.

Definition 1.4.14. First we consider the Bambozzi-Kremnizer spectrum $\mathrm{Spec}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,\Gamma,-},\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,-},\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,-}, \tag{1.4.73}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,\Gamma,-},\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,-},\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,-}, \tag{1.4.74}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,\Gamma,-},\mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,-},\mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,-}. \tag{1.4.75}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.76}$$

(1.4.77)

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.78}$$

(1.4.79)

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.4.80}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,-}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,-},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,-}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,-},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,-}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,-}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 1.4.15. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Namely A will still as above as a Banach ring over $\mathbb{Q}_p$. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_{*,\Gamma}, \widetilde{\nabla}_{*,\Gamma}, \widetilde{\Phi}_{*,\Gamma}, \widetilde{\Delta}^+_{*,\Gamma}, \widetilde{\nabla}^+_{*,\Gamma}, \widetilde{\Delta}^\dagger_{*,\Gamma}, \widetilde{\nabla}^\dagger_{*,\Gamma}, \widetilde{\Phi}^r_{*,\Gamma}, \widetilde{\Phi}^I_{*,\Gamma},$$

$$\breve{\Delta}_{*,\Gamma}, \breve{\nabla}_{*,\Gamma}, \breve{\Phi}_{*,\Gamma}, \breve{\Delta}^+_{*,\Gamma}, \breve{\nabla}^+_{*,\Gamma}, \breve{\Delta}^\dagger_{*,\Gamma}, \breve{\nabla}^\dagger_{*,\Gamma}, \breve{\Phi}^r_{*,\Gamma}, \breve{\Phi}^I_{*,\Gamma},$$

$$\Delta_{*,\Gamma}, \nabla_{*,\Gamma}, \Phi_{*,\Gamma}, \Delta^+_{*,\Gamma}, \nabla^+_{*,\Gamma}, \Delta^\dagger_{*,\Gamma}, \nabla^\dagger_{*,\Gamma}, \Phi^r_{*,\Gamma}, \Phi^I_{*,\Gamma},$$

with A. Then we have the notations:

$$\widetilde{\Delta}_{*,\Gamma,-}, \widetilde{\nabla}_{*,\Gamma,-}, \widetilde{\Phi}_{*,\Gamma,-}, \widetilde{\Delta}^+_{*,\Gamma,-}, \widetilde{\nabla}^+_{*,\Gamma,-}, \widetilde{\Delta}^\dagger_{*,\Gamma,-}, \widetilde{\nabla}^\dagger_{*,\Gamma,-}, \widetilde{\Phi}^r_{*,\Gamma,-}, \widetilde{\Phi}^I_{*,\Gamma,-},$$

$$\breve{\Delta}_{*,\Gamma,-}, \breve{\nabla}_{*,\Gamma,-}, \breve{\Phi}_{*,\Gamma,-}, \breve{\Delta}^+_{*,\Gamma,-}, \breve{\nabla}^+_{*,\Gamma,-}, \breve{\Delta}^\dagger_{*,\Gamma,-}, \breve{\nabla}^\dagger_{*,\Gamma,-}, \breve{\Phi}^r_{*,\Gamma,-}, \breve{\Phi}^I_{*,\Gamma,-},$$

$$\Delta_{*,\Gamma,-}, \nabla_{*,\Gamma,-}, \Phi_{*,\Gamma,-}, \Delta^+_{*,\Gamma,-}, \nabla^+_{*,\Gamma,-}, \Delta^\dagger_{*,\Gamma,-}, \nabla^\dagger_{*,\Gamma,-}, \Phi^r_{*,\Gamma,-}, \Phi^I_{*,\Gamma,-}.$$

Definition 1.4.16. First we consider the Clausen-Scholze spectrum $\mathrm{Spec}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,\Gamma,-}, \tag{1.4.81}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,-}, \tag{1.4.82}$$

$$\tag{1.4.83}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,-}, \breve{\nabla}_{*,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,\Gamma,-}, \tag{1.4.84}$$
$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,-}, \breve{\Phi}^I_{*,\Gamma,-}, \tag{1.4.85}$$
$$\tag{1.4.86}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,\Gamma,-}, \tag{1.4.87}$$
$$\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,-}, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,-}. \tag{1.4.88}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.89}$$
$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.90}$$
$$\tag{1.4.91}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.92}$$
$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.93}$$
$$\tag{1.4.94}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.95}$$
$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.4.96}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,-},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,\Gamma,-},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,-}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 1.4.17. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves}, \text{IndBanach}_{*} \tag{1.4.97}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{BK}}\widetilde{\Phi}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{1.4.98}$$

(1.4.99)

$$\text{Spec}^{\text{BK}}\breve{\Phi}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{1.4.100}$$

(1.4.101)

$$\text{Spec}^{\text{BK}}\Phi_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}. \tag{1.4.102}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{*,\Gamma,-}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{*,\Gamma,-},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^r_{*,\Gamma,-}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^I_{*,\Gamma,-},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\Phi^r_{*,\Gamma,-}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\Phi^I_{*,\Gamma,-}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^r_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^I_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\Phi^r_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\Phi^I_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}.$$

Definition 1.4.18. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves}, \text{Condensed}_{*} \tag{1.4.103}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{CS}}\widetilde{\Delta}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Phi}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^{+}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{1.4.104}$$

$$\text{Spec}^{\text{CS}}\widetilde{\nabla}^{+}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^{\dagger}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}^{\dagger}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{1.4.105}$$

(1.4.106)

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^{+}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.107}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^{+}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^{\dagger}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^{\dagger}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.108}$$

$$\tag{1.4.109}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^{+}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.110}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^{+}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^{\dagger}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^{\dagger}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.4.111}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,\Gamma,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,\Gamma,-},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,\Gamma,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{I}_{*,\Gamma,-},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{r}_{*,\Gamma,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{I}_{*,\Gamma,-}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^{I}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{r}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{I}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 1.4.19. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Condensed}_{*} \tag{1.4.112}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.113}$$

$$\tag{1.4.114}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.115}$$

$$\tag{1.4.116}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.117}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we

consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{*,\Gamma,-}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{*,\Gamma,-},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{*,\Gamma,-}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^I_{*,\Gamma,-},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{*,\Gamma,-}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{*,\Gamma,-}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{1.4.118}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{1.4.119}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 1.4.20. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{IndBanach}_*. \tag{1.4.120}$$

Definition 1.4.21. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves}, \text{Perfectcomplex}, \text{IndBanach}_* \tag{1.4.121}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{BK}}\widetilde{\Phi}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{1.4.122}$$

$$\tag{1.4.123}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.124}$$

$$ \tag{1.4.125}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.4.126}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,-},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,-},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,-}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 1.4.22. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\mathrm{Quasicoherentsheaves}, \mathrm{Perfectcomplex}, \mathrm{Condensed}_* \tag{1.4.127}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.128}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.129}$$

$$ \tag{1.4.130}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.131}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.132}$$

$$ \tag{1.4.133}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.134}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.4.135}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\, \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{*,\Gamma,-}, \underset{I}{\text{homotopycolimit}}\, \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{*,\Gamma,-},$$

$$\underset{r}{\text{homotopylimit}}\, \text{Spec}^{\text{CS}}\breve{\Phi}^r_{*,\Gamma,-}, \underset{I}{\text{homotopycolimit}}\, \text{Spec}^{\text{CS}}\breve{\Phi}^I_{*,\Gamma,-},$$

$$\underset{r}{\text{homotopylimit}}\, \text{Spec}^{\text{CS}}\Phi^r_{*,\Gamma,-}, \underset{I}{\text{homotopycolimit}}\, \text{Spec}^{\text{CS}}\Phi^I_{*,\Gamma,-}.$$

$$\underset{r}{\text{homotopylimit}}\, \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\, \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\, \text{Spec}^{\text{CS}}\breve{\Phi}^r_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\, \breve{\Phi}^I_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\, \text{Spec}^{\text{CS}}\Phi^r_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\, \text{Spec}^{\text{CS}}\Phi^I_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}.$$

Proposition 1.4.23. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Perfectcomplex, Condensed}_* \tag{1.4.136}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{CS}}\widetilde{\Phi}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{1.4.137}$$

(1.4.138)

$$\text{Spec}^{\text{CS}}\breve{\Phi}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{1.4.139}$$

(1.4.140)

$$\text{Spec}^{\text{CS}}\Phi_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{1.4.141}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\, \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{*,\Gamma,-}, \underset{I}{\text{homotopycolimit}}\, \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{*,\Gamma,-},$$

$$\underset{r}{\text{homotopylimit}}\, \text{Spec}^{\text{CS}}\breve{\Phi}^r_{*,\Gamma,-}, \underset{I}{\text{homotopycolimit}}\, \text{Spec}^{\text{CS}}\breve{\Phi}^I_{*,\Gamma,-},$$

$$\underset{r}{\text{homotopylimit}}\, \text{Spec}^{\text{CS}}\Phi^r_{*,\Gamma,-}, \underset{I}{\text{homotopycolimit}}\, \text{Spec}^{\text{CS}}\Phi^I_{*,\Gamma,-}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{1.4.142}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{1.4.143}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 1.4.24. Similar proposition holds for

$$\text{Quasicoherentsheaves, Perfectcomplex, IndBanach}_*. \tag{1.4.144}$$

1.4.3 Frobenius Quasicoherent Prestacks III: Deformation in $(\infty,1)$-Ind-Banach Rings

Definition 1.4.25. We now consider the pro-étale site of $\mathrm{Spa}\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$, denote that by $*$. To be more accurate we replace one component for Γ with the pro-étale site of $\mathrm{Spa}\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$. And we treat then all the functor to be prestacks for this site. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power:

$$\widetilde{\Delta}_{*,\Gamma},\widetilde{\nabla}_{*,\Gamma},\widetilde{\Phi}_{*,\Gamma},\widetilde{\Delta}^+_{*,\Gamma},\widetilde{\nabla}^+_{*,\Gamma},\widetilde{\Delta}^\dagger_{*,\Gamma},\widetilde{\nabla}^\dagger_{*,\Gamma},\widetilde{\Phi}^r_{*,\Gamma},\widetilde{\Phi}^I_{*,\Gamma},$$

$$\breve{\Delta}_{*,\Gamma},\breve{\nabla}_{*,\Gamma},\breve{\Phi}_{*,\Gamma},\breve{\Delta}^+_{*,\Gamma},\breve{\nabla}^+_{*,\Gamma},\breve{\Delta}^\dagger_{*,\Gamma},\breve{\nabla}^\dagger_{*,\Gamma},\breve{\Phi}^r_{*,\Gamma},\breve{\Phi}^I_{*,\Gamma},$$

$$\Delta_{*,\Gamma},\nabla_{*,\Gamma},\Phi_{*,\Gamma},\Delta^+_{*,\Gamma},\nabla^+_{*,\Gamma},\Delta^\dagger_{*,\Gamma},\nabla^\dagger_{*,\Gamma},\Phi^r_{*,\Gamma},\Phi^I_{*,\Gamma}.$$

We now consider the following rings with $\square$ being a homotopy colimit

$$\underset{i}{\mathrm{homotopycolimit}}\square_i \tag{1.4.145}$$

of $\mathbb{Q}_p\left\langle Y_1,...,Y_i\right\rangle, i=1,2,...$ in ∞-categories of simplicial ind-Banach rings in [BBBK]

$$\mathrm{SimplicialInd}-\mathrm{BanachRings}_{\mathbb{Q}_p} \tag{1.4.146}$$

or animated analytic condensed commutative algebras in [CS2]

$$\mathrm{SimplicialAnalyticCondensed}_{\mathbb{Q}_p}. \tag{1.4.147}$$

Taking the product we have:

$$\widetilde{\Phi}_{*,\Gamma,\square},\widetilde{\Phi}^r_{*,\Gamma,\square},\widetilde{\Phi}^I_{*,\Gamma,\square},$$

$$\breve{\Phi}_{*,\Gamma,\square},\breve{\Phi}^r_{*,\Gamma,\square},\breve{\Phi}^I_{*,\Gamma,\square},$$

$$\Phi_{*,\Gamma,\square},\Phi^r_{*,\Gamma,\square},\Phi^I_{*,\Gamma,\square}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma:=\mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty,1)$-categories of $(\infty,1)$-modules.

Definition 1.4.26. First we consider the Bambozzi-Kremnizer spectrum $\mathrm{Spec}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,\Gamma,\square},\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,\square},\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,\square}, \tag{1.4.148}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,\Gamma,\square},\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,\square},\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,\square}, \tag{1.4.149}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,\Gamma,\square},\mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,\square},\mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,\square}. \tag{1.4.150}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.151}$$

$$\tag{1.4.152}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.153}$$

$$\tag{1.4.154}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.4.155}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,\square}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 1.4.27. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Namely A will still as above as a Banach ring over $\mathbb{Q}_p$. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_{*,\Gamma}, \widetilde{\nabla}_{*,\Gamma}, \widetilde{\Phi}_{*,\Gamma}, \widetilde{\Delta}^+_{*,\Gamma}, \widetilde{\nabla}^+_{*,\Gamma}, \widetilde{\Delta}^\dagger_{*,\Gamma}, \widetilde{\nabla}^\dagger_{*,\Gamma}, \widetilde{\Phi}^r_{*,\Gamma}, \widetilde{\Phi}^I_{*,\Gamma},$$

$$\breve{\Delta}_{*,\Gamma}, \breve{\nabla}_{*,\Gamma}, \breve{\Phi}_{*,\Gamma}, \breve{\Delta}^+_{*,\Gamma}, \breve{\nabla}^+_{*,\Gamma}, \breve{\Delta}^\dagger_{*,\Gamma}, \breve{\nabla}^\dagger_{*,\Gamma}, \breve{\Phi}^r_{*,\Gamma}, \breve{\Phi}^I_{*,\Gamma},$$

$$\Delta_{*,\Gamma}, \nabla_{*,\Gamma}, \Phi_{*,\Gamma}, \Delta^+_{*,\Gamma}, \nabla^+_{*,\Gamma}, \Delta^\dagger_{*,\Gamma}, \nabla^\dagger_{*,\Gamma}, \Phi^r_{*,\Gamma}, \Phi^I_{*,\Gamma},$$

with A. Then we have the notations:

$$\widetilde{\Delta}_{*,\Gamma,\square}, \widetilde{\nabla}_{*,\Gamma,\square}, \widetilde{\Phi}_{*,\Gamma,\square}, \widetilde{\Delta}^+_{*,\Gamma,\square}, \widetilde{\nabla}^+_{*,\Gamma,\square}, \widetilde{\Delta}^\dagger_{*,\Gamma,\square}, \widetilde{\nabla}^\dagger_{*,\Gamma,\square}, \widetilde{\Phi}^r_{*,\Gamma,\square}, \widetilde{\Phi}^I_{*,\Gamma,\square},$$

$$\breve{\Delta}_{*,\Gamma,\square}, \breve{\nabla}_{*,\Gamma,\square}, \breve{\Phi}_{*,\Gamma,\square}, \breve{\Delta}^+_{*,\Gamma,\square}, \breve{\nabla}^+_{*,\Gamma,\square}, \breve{\Delta}^\dagger_{*,\Gamma,\square}, \breve{\nabla}^\dagger_{*,\Gamma,\square}, \breve{\Phi}^r_{*,\Gamma,\square}, \breve{\Phi}^I_{*,\Gamma,\square},$$

$$\Delta_{*,\Gamma,\square}, \nabla_{*,\Gamma,\square}, \Phi_{*,\Gamma,\square}, \Delta^+_{*,\Gamma,\square}, \nabla^+_{*,\Gamma,\square}, \Delta^\dagger_{*,\Gamma,\square}, \nabla^\dagger_{*,\Gamma,\square}, \Phi^r_{*,\Gamma,\square}, \Phi^I_{*,\Gamma,\square}.$$

Definition 1.4.28. First we consider the Clausen-Scholze spectrum $\mathrm{Spec}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,\Gamma,\square}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,\Gamma,\square}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,\square}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,\Gamma,\square}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,\Gamma,\square}, \tag{1.4.156}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,\Gamma,\square}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,\Gamma,\square}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,\square}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,\square}, \tag{1.4.157}$$

$$\tag{1.4.158}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,\square}, \breve{\nabla}_{*,\Gamma,\square}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,\square}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,\Gamma,\square}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,\Gamma,\square}, \tag{1.4.159}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,\Gamma,\square}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,\Gamma,\square}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,\square}, \breve{\Phi}^I_{*,\Gamma,\square}, \tag{1.4.160}$$

$$\tag{1.4.161}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\Gamma,\square}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\Gamma,\square}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,\square}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,\Gamma,\square}, \mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,\Gamma,\square}, \tag{1.4.162}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,\Gamma,\square}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,\Gamma,\square}, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,\square}, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,\square}. \tag{1.4.163}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.164}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.165}$$

$$\tag{1.4.166}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.167}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.168}$$

$$\tag{1.4.169}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.170}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.4.171}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,\square}, \underset{I}{\mathrm{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,\square}, \underset{I}{\mathrm{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,\square}, \underset{I}{\mathrm{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,\square}.$$

$$\underset{r}{\mathrm{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\, \breve{\Phi}^I_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 1.4.29. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves, IndBanach}_{*} \tag{1.4.172}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{BK}}\widetilde{\Phi}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{1.4.173}$$
$$\tag{1.4.174}$$

$$\text{Spec}^{\text{BK}}\breve{\Phi}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{1.4.175}$$
$$\tag{1.4.176}$$

$$\text{Spec}^{\text{BK}}\Phi_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}. \tag{1.4.177}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\text{homotopylimit}_r \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{*,\Gamma,\square}, \text{homotopycolimit}_I \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{*,\Gamma,\square},$$

$$\text{homotopylimit}_r \text{Spec}^{\text{BK}}\breve{\Phi}^r_{*,\Gamma,\square}, \text{homotopycolimit}_I \text{Spec}^{\text{BK}}\breve{\Phi}^I_{*,\Gamma,\square},$$

$$\text{homotopylimit}_r \text{Spec}^{\text{BK}}\Phi^r_{*,\Gamma,\square}, \text{homotopycolimit}_I \text{Spec}^{\text{BK}}\Phi^I_{*,\Gamma,\square}.$$

$$\text{homotopylimit}_r \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \text{homotopycolimit}_I \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\text{homotopylimit}_r \text{Spec}^{\text{BK}}\breve{\Phi}^r_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \text{homotopycolimit}_I \text{Spec}^{\text{BK}}\breve{\Phi}^I_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\text{homotopylimit}_r \text{Spec}^{\text{BK}}\Phi^r_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \text{homotopycolimit}_I \text{Spec}^{\text{BK}}\Phi^I_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}.$$

Definition 1.4.30. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Condensed}_{*} \tag{1.4.178}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{CS}}\widetilde{\Delta}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Phi}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^{+}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{1.4.179}$$
$$\text{Spec}^{\text{CS}}\widetilde{\nabla}^{+}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^{\dagger}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}^{\dagger}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{1.4.180}$$
$$\tag{1.4.181}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\breve{\nabla}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.182}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.183}$$

$$\tag{1.4.184}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.185}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.4.186}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r,\bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\,\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,\square},\underset{I}{\mathrm{homotopycolimit}}\,\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\,\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,\square},\underset{I}{\mathrm{homotopycolimit}}\,\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\,\mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,\square},\underset{I}{\mathrm{homotopycolimit}}\,\mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,\square}.$$

$$\underset{r}{\mathrm{homotopylimit}}\,\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\underset{I}{\mathrm{homotopycolimit}}\,\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\,\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\underset{I}{\mathrm{homotopycolimit}}\,\breve{\Phi}^I_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\,\mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},\underset{I}{\mathrm{homotopycolimit}}\,\mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 1.4.31. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves},\mathrm{Condensed}_* \tag{1.4.187}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.188}$$

$$\tag{1.4.189}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.190}$$

$$\tag{1.4.191}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.192}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we

consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,\square}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,\square},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,\square}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,\Gamma,\square},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,\square}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,\square}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{1.4.193}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{1.4.194}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 1.4.32. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{IndBanach}_*. \tag{1.4.195}$$

Definition 1.4.33. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves}, \text{Perfectcomplex}, \text{IndBanach}_* \tag{1.4.196}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.197}$$

$$\tag{1.4.198}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.199}$$

$$\tag{1.4.200}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.4.201}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,\square}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 1.4.34. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\mathrm{Quasicoherentsheaves}, \mathrm{Perfectcomplex}, \mathrm{Condensed}_* \tag{1.4.202}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.203}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.204}$$

$$\tag{1.4.205}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.206}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.207}$$

$$\tag{1.4.208}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.4.209}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.4.210}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{*,\Gamma,\square}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{*,\Gamma,\square},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{*,\Gamma,\square}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^I_{*,\Gamma,\square},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{*,\Gamma,\square}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{*,\Gamma,\square}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}.$$

Proposition 1.4.35. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Perfectcomplex, Condensed}_* \tag{1.4.211}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{CS}}\widetilde{\Phi}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{1.4.212}$$

(1.4.213)

$$\text{Spec}^{\text{CS}}\breve{\Phi}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{1.4.214}$$

(1.4.215)

$$\text{Spec}^{\text{CS}}\Phi_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{1.4.216}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{*,\Gamma,\square}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{*,\Gamma,\square},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{*,\Gamma,\square}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^I_{*,\Gamma,\square},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{*,\Gamma,\square}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{*,\Gamma,\square}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{1.4.217}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{1.4.218}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 1.4.36. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{Perfectcomplex}, \text{IndBanach}_*. \tag{1.4.219}$$

1.5 Univariate Hodge Iwasawa Prestacks

This chapter follows closely [T1], [T2], [T3], [KPX], [KP], [KL1], [KL2], [BK], [BBBK], [BBM], [KKM], [CS1], [CS2], [LBV].

1.5.1 Frobenius Quasicoherent Prestacks I

Definition 1.5.1. We now consider the pro-étale site of $\text{Spa}\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$, denote that by $*$. To be more accurate we replace one component for Γ with the pro-étale site of $\text{Spa}\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$. And we treat then all the functor to be prestacks for this site. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power[8]:

$$\widetilde{\Delta}_*, \widetilde{\nabla}_*, \widetilde{\Phi}_*, \widetilde{\Delta}^+_*, \widetilde{\nabla}^+_*, \widetilde{\Delta}^\dagger_*, \widetilde{\nabla}^\dagger_*, \widetilde{\Phi}^r_*, \widetilde{\Phi}^I_*,$$

$$\breve{\Delta}_*, \breve{\nabla}_*, \breve{\Phi}_*, \breve{\Delta}^+_*, \breve{\nabla}^+_*, \breve{\Delta}^\dagger_*, \breve{\nabla}^\dagger_*, \breve{\Phi}^r_*, \breve{\Phi}^I_*,$$

$$\Delta_*, \nabla_*, \Phi_*, \Delta^+_*, \nabla^+_*, \Delta^\dagger_*, \nabla^\dagger_*, \Phi^r_*, \Phi^I_*.$$

[8] Here $|\Gamma| = 1$.

We now consider the following rings with A being a Banach ring over $\mathbb{Q}_p$. Taking the product we have:

$$\widetilde{\Phi}_{*,A}, \widetilde{\Phi}^r_{*,A}, \widetilde{\Phi}^I_{*,A},$$

$$\breve{\Phi}_{*,A}, \breve{\Phi}^r_{*,A}, \breve{\Phi}^I_{*,A},$$

$$\Phi_{*,A}, \Phi^r_{*,A}, \Phi^I_{*,A}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma := \mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty,1)$-categories of $(\infty,1)$-modules.

Definition 1.5.2. First we consider the Bambozzi-Kremnizer spectrum $\mathrm{Spec}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,A}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,A}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,A}, \tag{1.5.1}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,A}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,A}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,A}, \tag{1.5.2}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,A}, \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,A}, \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,A}. \tag{1.5.3}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.4}$$

(1.5.5)

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.6}$$

(1.5.7)

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.5.8}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,A}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 1.5.3. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Namely A will still as above as a Banach ring over $\mathbb{Q}_p$. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_*, \widetilde{\nabla}_*, \widetilde{\Phi}_*, \widetilde{\Delta}^+_*, \widetilde{\nabla}^+_*, \widetilde{\Delta}^\dagger_*, \widetilde{\nabla}^\dagger_*, \widetilde{\Phi}^r_*, \widetilde{\Phi}^I_*,$$

$$\breve{\Delta}_*, \breve{\nabla}_*, \breve{\Phi}_*, \breve{\Delta}^+_*, \breve{\nabla}^+_*, \breve{\Delta}^\dagger_*, \breve{\nabla}^\dagger_*, \breve{\Phi}^r_*, \breve{\Phi}^I_*,$$

$$\Delta_*, \nabla_*, \Phi_*, \Delta^+_*, \nabla^+_*, \Delta^\dagger_*, \nabla^\dagger_*, \Phi^r_*, \Phi^I_*,$$

with A. Then we have the notations:

$$\widetilde{\Delta}_{*,A}, \widetilde{\nabla}_{*,A}, \widetilde{\Phi}_{*,A}, \widetilde{\Delta}^+_{*,A}, \widetilde{\nabla}^+_{*,A}, \widetilde{\Delta}^\dagger_{*,A}, \widetilde{\nabla}^\dagger_{*,A}, \widetilde{\Phi}^r_{*,A}, \widetilde{\Phi}^I_{*,A},$$

$$\breve{\Delta}_{*,A}, \breve{\nabla}_{*,A}, \breve{\Phi}_{*,A}, \breve{\Delta}^+_{*,A}, \breve{\nabla}^+_{*,A}, \breve{\Delta}^\dagger_{*,A}, \breve{\nabla}^\dagger_{*,A}, \breve{\Phi}^r_{*,A}, \breve{\Phi}^I_{*,A},$$

$$\Delta_{*,A}, \nabla_{*,A}, \Phi_{*,A}, \Delta^+_{*,A}, \nabla^+_{*,A}, \Delta^\dagger_{*,A}, \nabla^\dagger_{*,A}, \Phi^r_{*,A}, \Phi^I_{*,A}.$$

Definition 1.5.4. First we consider the Clausen-Scholze spectrum $\mathrm{Spec}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,A}, \tag{1.5.9}$$
$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,A}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,A}, \tag{1.5.10}$$
$$\tag{1.5.11}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,A}, \breve{\nabla}_{*,A}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,A}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,A}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,A}, \tag{1.5.12}$$
$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,A}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,A}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,A}, \breve{\Phi}^I_{*,A}, \tag{1.5.13}$$
$$\tag{1.5.14}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,A}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,A}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,A}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,A}, \mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,A}, \tag{1.5.15}$$
$$\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,A}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,A}, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,A}, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,A}. \tag{1.5.16}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.17}$$
$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.18}$$
$$\tag{1.5.19}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,A}/\mathrm{Fro}^\mathbb{Z}, \breve{\nabla}_{*,A}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,A}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,A}/\mathrm{Fro}^\mathbb{Z}, \tag{1.5.20}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,A}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,A}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,A}/\mathrm{Fro}^\mathbb{Z}, \tag{1.5.21}$$

$$\tag{1.5.22}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,A}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,A}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,A}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,A}/\mathrm{Fro}^\mathbb{Z}, \tag{1.5.23}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,A}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,A}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,A}/\mathrm{Fro}^\mathbb{Z}. \tag{1.5.24}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,A}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,A}/\mathrm{Fro}^\mathbb{Z}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,A}/\mathrm{Fro}^\mathbb{Z},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,A}/\mathrm{Fro}^\mathbb{Z}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{*,A}/\mathrm{Fro}^\mathbb{Z},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,A}/\mathrm{Fro}^\mathbb{Z}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,A}/\mathrm{Fro}^\mathbb{Z}.$$

Definition 1.5.5. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\mathrm{Quasicoherentpresheaves}, \mathrm{IndBanach}_* \tag{1.5.25}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,A}/\mathrm{Fro}^\mathbb{Z}, \tag{1.5.26}$$

$$\tag{1.5.27}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,A}/\mathrm{Fro}^\mathbb{Z}, \tag{1.5.28}$$

$$\tag{1.5.29}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,A}/\mathrm{Fro}^\mathbb{Z}. \tag{1.5.30}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,A}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 1.5.6. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\mathrm{Quasicoherentsheaves}, \mathrm{Condensed}_{*} \tag{1.5.31}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.32}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.33}$$

$$\tag{1.5.34}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.35}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.36}$$

$$\tag{1.5.37}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.38}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.5.39}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,A}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{*,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 1.5.7. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves,Condensed}_{*} \tag{1.5.40}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.41}$$

$$\tag{1.5.42}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.43}$$

$$\tag{1.5.44}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.45}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,A}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{*,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{1.5.46}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{1.5.47}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 1.5.8. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{IndBanach}_{*}. \tag{1.5.48}$$

Definition 1.5.9. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves}, \text{Perfectcomplex}, \text{IndBanach}_{*} \tag{1.5.49}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.50}$$

$$\tag{1.5.51}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.52}$$

$$\tag{1.5.53}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.5.54}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,A}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,A},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,A}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,A},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,A}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,A}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 1.5.10. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves}, \text{Perfectcomplex}, \text{Condensed}_{*} \tag{1.5.55}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^{+}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.56}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^{+}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^{\dagger}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^{\dagger}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.57}$$

$$\tag{1.5.58}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^{+}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.59}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^{+}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^{\dagger}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^{\dagger}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.60}$$

$$\tag{1.5.61}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^{+}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.62}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^{+}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^{\dagger}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^{\dagger}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.5.63}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{I}_{*,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{r}_{*,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{I}_{*,A}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^{I}_{*,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{r}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{I}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 1.5.11. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Perfectcomplex}, \mathrm{Condensed}_{*} \tag{1.5.64}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.65}$$

$$\tag{1.5.66}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.67}$$

$$\tag{1.5.68}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.69}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,A},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,A}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,A}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{*,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{1.5.70}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{1.5.71}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 1.5.12. Similar proposition holds for

$$\mathrm{Quasicoherentsheaves}, \mathrm{Perfectcomplex}, \mathrm{IndBanach}_*. \tag{1.5.72}$$

1.5.2 Frobenius Quasicoherent Prestacks II: Deformation in Banach Rings

Definition 1.5.13. We now consider the pro-étale site of $\mathrm{Spa}\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$, denote that by $*$. To be more accurate we replace one component for Γ with the pro-étale site of $\mathrm{Spa}\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$. And we treat then all the functor to be prestacks for this site. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power[9]:

$$\widetilde{\Delta}_*, \widetilde{\nabla}_*, \widetilde{\Phi}_*, \widetilde{\Delta}^+_*, \widetilde{\nabla}^+_*, \widetilde{\Delta}^\dagger_*, \widetilde{\nabla}^\dagger_*, \widetilde{\Phi}^r_*, \widetilde{\Phi}^I_*,$$

$$\breve{\Delta}_*, \breve{\nabla}_*, \breve{\Phi}_*, \breve{\Delta}^+_*, \breve{\nabla}^+_*, \breve{\Delta}^\dagger_*, \breve{\nabla}^\dagger_*, \breve{\Phi}^r_*, \breve{\Phi}^I_*,$$

$$\Delta_*, \nabla_*, \Phi_*, \Delta^+_*, \nabla^+_*, \Delta^\dagger_*, \nabla^\dagger_*, \Phi^r_*, \Phi^I_*.$$

We now consider the following rings with $-$ being any deforming Banach ring over $\mathbb{Q}_p$. Taking the product we have:

$$\widetilde{\Phi}_{*,-}, \widetilde{\Phi}^r_{*,-}, \widetilde{\Phi}^I_{*,-},$$

$$\breve{\Phi}_{*,-}, \breve{\Phi}^r_{*,-}, \breve{\Phi}^I_{*,-},$$

$$\Phi_{*,-}, \Phi^r_{*,-}, \Phi^I_{*,-}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma := \mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty,1)$-categories of $(\infty,1)$-modules.

Definition 1.5.14. First we consider the Bambozzi-Kremnizer spectrum $\mathrm{Spec}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,-}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,-}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,-}, \tag{1.5.73}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,-}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,-}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,-}, \tag{1.5.74}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,-}, \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,-}, \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,-}. \tag{1.5.75}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.76}$$

$$\tag{1.5.77}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.78}$$

$$\tag{1.5.79}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,-}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.5.80}$$

[9]Here $|\Gamma| = 1$.

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,-}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,-},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,-}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,-},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,-}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,-}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 1.5.15. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Namely A will still as above as a Banach ring over $\mathbb{Q}_p$. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_*, \widetilde{\nabla}_*, \widetilde{\Phi}_*, \widetilde{\Delta}^+_*, \widetilde{\nabla}^+_*, \widetilde{\Delta}^\dagger_*, \widetilde{\nabla}^\dagger_*, \widetilde{\Phi}^r_*, \widetilde{\Phi}^I_*,$$

$$\breve{\Delta}_*, \breve{\nabla}_*, \breve{\Phi}_*, \breve{\Delta}^+_*, \breve{\nabla}^+_*, \breve{\Delta}^\dagger_*, \breve{\nabla}^\dagger_*, \breve{\Phi}^r_*, \breve{\Phi}^I_*,$$

$$\Delta_*, \nabla_*, \Phi_*, \Delta^+_*, \nabla^+_*, \Delta^\dagger_*, \nabla^\dagger_*, \Phi^r_*, \Phi^I_*,$$

with A. Then we have the notations:

$$\widetilde{\Delta}_{*,-}, \widetilde{\nabla}_{*,-}, \widetilde{\Phi}_{*,-}, \widetilde{\Delta}^+_{*,-}, \widetilde{\nabla}^+_{*,-}, \widetilde{\Delta}^\dagger_{*,-}, \widetilde{\nabla}^\dagger_{*,-}, \widetilde{\Phi}^r_{*,-}, \widetilde{\Phi}^I_{*,-},$$

$$\breve{\Delta}_{*,-}, \breve{\nabla}_{*,-}, \breve{\Phi}_{*,-}, \breve{\Delta}^+_{*,-}, \breve{\nabla}^+_{*,-}, \breve{\Delta}^\dagger_{*,-}, \breve{\nabla}^\dagger_{*,-}, \breve{\Phi}^r_{*,-}, \breve{\Phi}^I_{*,-},$$

$$\Delta_{*,-}, \nabla_{*,-}, \Phi_{*,-}, \Delta^+_{*,-}, \nabla^+_{*,-}, \Delta^\dagger_{*,-}, \nabla^\dagger_{*,-}, \Phi^r_{*,-}, \Phi^I_{*,-}.$$

Definition 1.5.16. First we consider the Clausen-Scholze spectrum $\mathrm{Spec}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,-}, \tag{1.5.81}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,-}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,-}, \tag{1.5.82}$$

$$\tag{1.5.83}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,-}, \breve{\nabla}_{*,-}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,-}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,-}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,-}, \tag{1.5.84}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,-}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,-}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,-}, \breve{\Phi}^I_{*,-}, \tag{1.5.85}$$

$$\tag{1.5.86}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,-}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,-}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,-}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,-}, \mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,-}, \tag{1.5.87}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,-}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,-}, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,-}, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,-}. \tag{1.5.88}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.89}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.90}$$

$$\tag{1.5.91}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.92}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.93}$$

$$\tag{1.5.94}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.95}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,-}/\mathrm{Fro}^{\mathbb{Z}}. \tag{1.5.96}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,-},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,-},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,-}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{*,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 1.5.17. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves, IndBanach}_* \tag{1.5.97}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{BK}}\widetilde{\Phi}_{*,-}/\text{Fro}^{\mathbb{Z}}, \tag{1.5.98}$$
$$\tag{1.5.99}$$

$$\text{Spec}^{\text{BK}}\breve{\Phi}_{*,-}/\text{Fro}^{\mathbb{Z}}, \tag{1.5.100}$$
$$\tag{1.5.101}$$

$$\text{Spec}^{\text{BK}}\Phi_{*,-}/\text{Fro}^{\mathbb{Z}}. \tag{1.5.102}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{*,-}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{*,-},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^r_{*,-}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^I_{*,-},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\Phi^r_{*,-}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\Phi^I_{*,-}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{*,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{*,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^r_{*,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^I_{*,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\Phi^r_{*,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\Phi^I_{*,-}/\text{Fro}^{\mathbb{Z}}.$$

Definition 1.5.18. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Condensed}_* \tag{1.5.103}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{CS}}\widetilde{\Delta}_{*,-}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}_{*,-}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Phi}_{*,-}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^+_{*,-}/\text{Fro}^{\mathbb{Z}}, \tag{1.5.104}$$
$$\text{Spec}^{\text{CS}}\widetilde{\nabla}^+_{*,-}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^\dagger_{*,-}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}^\dagger_{*,-}/\text{Fro}^{\mathbb{Z}}, \tag{1.5.105}$$
$$\tag{1.5.106}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,-}/\mathrm{Fro}^\mathbb{Z}, \breve{\nabla}_{*,-}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,-}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,-}/\mathrm{Fro}^\mathbb{Z}, \tag{1.5.107}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,-}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,-}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,-}/\mathrm{Fro}^\mathbb{Z}, \tag{1.5.108}$$

$$\tag{1.5.109}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,-}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,-}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,-}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,-}/\mathrm{Fro}^\mathbb{Z}, \tag{1.5.110}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,-}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,-}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,-}/\mathrm{Fro}^\mathbb{Z}. \tag{1.5.111}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,-},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,-},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,-}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,-}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,-}/\mathrm{Fro}^\mathbb{Z}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,-}/\mathrm{Fro}^\mathbb{Z},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,-}/\mathrm{Fro}^\mathbb{Z}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{*,-}/\mathrm{Fro}^\mathbb{Z},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,-}/\mathrm{Fro}^\mathbb{Z}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,-}/\mathrm{Fro}^\mathbb{Z}.$$

Proposition 1.5.19. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}_{,\mathrm{Condensed}_*} \tag{1.5.112}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,-}/\mathrm{Fro}^\mathbb{Z}, \tag{1.5.113}$$

$$\tag{1.5.114}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,-}/\mathrm{Fro}^\mathbb{Z}, \tag{1.5.115}$$

$$\tag{1.5.116}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{*,-}/\mathrm{Fro}^\mathbb{Z}, \tag{1.5.117}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we

consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,-}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,-},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,-}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,-},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,-}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,-}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{*,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{1.5.118}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{1.5.119}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 1.5.20. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{IndBanach}_*. \tag{1.5.120}$$

Definition 1.5.21. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves}, \text{Perfectcomplex}, \text{IndBanach}_* \tag{1.5.121}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.122}$$

$$\tag{1.5.123}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,-}/\mathrm{Fro}^{\mathbb{Z}},\tag{1.5.124}$$

$$\tag{1.5.125}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,-}/\mathrm{Fro}^{\mathbb{Z}}.\tag{1.5.126}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\mathrm{homotopylimit}_r\,\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,-}, \mathrm{homotopycolimit}_I\,\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,-},$$

$$\mathrm{homotopylimit}_r\,\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,-}, \mathrm{homotopycolimit}_I\,\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,-},$$

$$\mathrm{homotopylimit}_r\,\mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,-}, \mathrm{homotopycolimit}_I\,\mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,-}.$$

$$\mathrm{homotopylimit}_r\,\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{homotopycolimit}_I\,\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\mathrm{homotopylimit}_r\,\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{homotopycolimit}_I\,\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\mathrm{homotopylimit}_r\,\mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{homotopycolimit}_I\,\mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 1.5.22. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\mathrm{Quasicoherentsheaves, Perfectcomplex, Condensed}_{*}\tag{1.5.127}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,-}/\mathrm{Fro}^{\mathbb{Z}},\tag{1.5.128}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,-}/\mathrm{Fro}^{\mathbb{Z}},\tag{1.5.129}$$

$$\tag{1.5.130}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,-}/\mathrm{Fro}^{\mathbb{Z}},\tag{1.5.131}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,-}/\mathrm{Fro}^{\mathbb{Z}},\tag{1.5.132}$$

$$\tag{1.5.133}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,-}/\mathrm{Fro}^{\mathbb{Z}},\tag{1.5.134}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,-}/\mathrm{Fro}^{\mathbb{Z}}.\tag{1.5.135}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\mathrm{homotopylimit}_r\,\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,-}, \mathrm{homotopycolimit}_I\,\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,-},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,-}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,-},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,-}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,-}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{*,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 1.5.23. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Perfectcomplex, Condensed}_* \tag{1.5.136}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.137}$$

$$\tag{1.5.138}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.139}$$

$$\tag{1.5.140}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{1.5.141}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,-}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,-},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,-}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,-},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,-}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,-}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{*,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\,\mathrm{Spec}^{\mathrm{CS}}\Phi^{r}_{*,-}/\mathrm{Fro}^{\mathbb{Z}},\underset{I}{\text{homotopycolimit}}\,\mathrm{Spec}^{\mathrm{CS}}\Phi^{I}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\,M_r, \tag{1.5.142}$$

$$\underset{I}{\text{homotopylimit}}\,M_I, \tag{1.5.143}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 1.5.24. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{Perfectcomplex}, \text{IndBanach}_*. \tag{1.5.144}$$

Chapter 2

Deformation

2.1 Multivariate Hodge Iwasawa Modules by Deformation

This chapter follows closely [T1], [T2], [T3], [T4], [T5], [T6], [KPX], [KP], [KL1], [KL2], [BK], [BBBK], [BBM], [KKM], [CS1], [CS2], [CKZ], [PZ], [BCM], [LBV].

Remark 2.1.1. In the following chapters, we remind the readers of the fact that the notations for the deformation $A, -, \square$ in our following discussion will mean different thing, the deformation with respect to $A, -, \square$ will happen along the structure sheaves $\mathcal{O}$ directly. The ∞-descent results in [BK], [BBBK], [BBM], [KKM], [CS1], [CS2] will guarantee that the deformed sheaves are still quasicoherent sheaves over $\mathcal{O}$.

2.1.1 Frobenius Quasicoherent Modules I

Definition 2.1.2. First we consider the Bambozzi-Kremnizer spectrum $\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,A}, \tag{2.1.1}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}_{\psi,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,A}, \tag{2.1.2}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi_{\psi,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,A}. \tag{2.1.3}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.4}$$

$$\tag{2.1.5}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.6}$$

$$\tag{2.1.7}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.1.8}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,A}.$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 2.1.3. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Namely A will still as above as a Banach ring over $\mathbb{Q}_p$. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_{\psi,\Gamma}, \widetilde{\nabla}_{\psi,\Gamma}, \widetilde{\Phi}_{\psi,\Gamma}, \widetilde{\Delta}^+_{\psi,\Gamma}, \widetilde{\nabla}^+_{\psi,\Gamma}, \widetilde{\Delta}^\dagger_{\psi,\Gamma}, \widetilde{\nabla}^\dagger_{\psi,\Gamma}, \widetilde{\Phi}^r_{\psi,\Gamma}, \widetilde{\Phi}^I_{\psi,\Gamma},$$

$$\breve{\Delta}_{\psi,\Gamma}, \breve{\nabla}_{\psi,\Gamma}, \breve{\Phi}_{\psi,\Gamma}, \breve{\Delta}^+_{\psi,\Gamma}, \breve{\nabla}^+_{\psi,\Gamma}, \breve{\Delta}^\dagger_{\psi,\Gamma}, \breve{\nabla}^\dagger_{\psi,\Gamma}, \breve{\Phi}^r_{\psi,\Gamma}, \breve{\Phi}^I_{\psi,\Gamma},$$

$$\Delta_{\psi,\Gamma}, \nabla_{\psi,\Gamma}, \Phi_{\psi,\Gamma}, \Delta^+_{\psi,\Gamma}, \nabla^+_{\psi,\Gamma}, \Delta^\dagger_{\psi,\Gamma}, \nabla^\dagger_{\psi,\Gamma}, \Phi^r_{\psi,\Gamma}, \Phi^I_{\psi,\Gamma},$$

with A. Then we have the notations:

$$\widetilde{\Delta}_{\psi,\Gamma,A}, \widetilde{\nabla}_{\psi,\Gamma,A}, \widetilde{\Phi}_{\psi,\Gamma,A}, \widetilde{\Delta}^+_{\psi,\Gamma,A}, \widetilde{\nabla}^+_{\psi,\Gamma,A}, \widetilde{\Delta}^\dagger_{\psi,\Gamma,A}, \widetilde{\nabla}^\dagger_{\psi,\Gamma,A}, \widetilde{\Phi}^r_{\psi,\Gamma,A}, \widetilde{\Phi}^I_{\psi,\Gamma,A},$$

$$\breve{\Delta}_{\psi,\Gamma,A}, \breve{\nabla}_{\psi,\Gamma,A}, \breve{\Phi}_{\psi,\Gamma,A}, \breve{\Delta}^+_{\psi,\Gamma,A}, \breve{\nabla}^+_{\psi,\Gamma,A}, \breve{\Delta}^\dagger_{\psi,\Gamma,A}, \breve{\nabla}^\dagger_{\psi,\Gamma,A}, \breve{\Phi}^r_{\psi,\Gamma,A}, \breve{\Phi}^I_{\psi,\Gamma,A},$$

$$\Delta_{\psi,\Gamma,A}, \nabla_{\psi,\Gamma,A}, \Phi_{\psi,\Gamma,A}, \Delta^+_{\psi,\Gamma,A}, \nabla^+_{\psi,\Gamma,A}, \Delta^\dagger_{\psi,\Gamma,A}, \nabla^\dagger_{\psi,\Gamma,A}, \Phi^r_{\psi,\Gamma,A}, \Phi^I_{\psi,\Gamma,A}.$$

Definition 2.1.4. First we consider the Clausen-Scholze spectrum $\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,A}, \tag{2.1.9}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,A}, \tag{2.1.10}$$

$$\tag{2.1.11}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,A}, \breve{\nabla}_{\psi,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,A}, \tag{2.1.12}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,A}, \breve{\Phi}^I_{\psi,\Gamma,A}, \tag{2.1.13}$$

$$\tag{2.1.14}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta_{\psi,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla_{\psi,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{\psi,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,A}, \tag{2.1.15}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,A}. \tag{2.1.16}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.17}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.18}$$

$$\tag{2.1.19}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.20}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.21}$$

$$\tag{2.1.22}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.23}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.1.24}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\mathrm{homotopylimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopycolimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\mathrm{homotopylimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{\psi,\Gamma,A}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{\psi,\Gamma,A}.$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^r_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \breve{\Phi}^I_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}.$$

Definition 2.1.5. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves}, \text{IndBanach}_* \tag{2.1.25}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.26}$$

(2.1.27)

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.28}$$

(2.1.29)

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}. \tag{2.1.30}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^r_{\psi,\Gamma,A}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^I_{\psi,\Gamma,A}.$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^r_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^I_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^r_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^I_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}.$$

Definition 2.1.6. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves}, \text{Condensed}_{*} \tag{2.1.31}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}^{+}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.32}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}^{+}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}^{\dagger}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}^{\dagger}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.33}$$

$$\tag{2.1.34}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}^{+}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.35}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\nabla}^{+}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}^{\dagger}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\nabla}^{\dagger}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.36}$$

$$\tag{2.1.37}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta^{+}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.38}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla^{+}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta^{\dagger}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla^{\dagger}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}. \tag{2.1.39}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{r}_{\psi,\Gamma,A}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{I}_{\psi,\Gamma,A},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^{r}_{\psi,\Gamma,A}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^{I}_{\psi,\Gamma,A},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{r}_{\psi,\Gamma,A}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{I}_{\psi,\Gamma,A}.$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{r}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{I}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^{r}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \breve{\Phi}^{I}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{r}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{I}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}.$$

Proposition 2.1.7. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves}, \text{Condensed}_{*} \tag{2.1.40}$$

where $*$ is one of the following spaces:

$$\mathcal{O}_{\text{Spec}}{}^{\text{CS}}\widetilde{\Phi}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.41}$$

$$\tag{2.1.42}$$

$$\mathcal{O}_{\text{Spec}}{}^{\text{CS}}\breve{\Phi}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.43}$$

$$\tag{2.1.44}$$

$$\mathcal{O}_{\text{Spec}}{}^{\text{CS}}\Phi_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.45}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}}\ \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\text{homotopylimit}}\ \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\text{homotopycolimit}}\ \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\breve{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\text{homotopylimit}}\ \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\breve{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\text{homotopycolimit}}\ \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\Phi^r_{\psi,\Gamma,A}, \underset{I}{\text{homotopylimit}}\ \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\Phi^I_{\psi,\Gamma,A}.$$

$$\underset{r}{\text{homotopycolimit}}\ \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\breve{\Phi}^r_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \breve{\Phi}^I_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\Phi^r_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\Phi^I_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{2.1.46}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{2.1.47}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 2.1.8. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{IndBanach}_{*}. \tag{2.1.48}$$

Definition 2.1.9. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves}, \text{Perfectcomplex}, \text{IndBanach}_{*} \tag{2.1.49}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{O}{}^{\text{BK}}\widetilde{\Phi}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.50}$$

$$\tag{2.1.51}$$

$$\underset{\text{Spec}}{O}{}^{\text{BK}}\breve{\Phi}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.52}$$

$$\tag{2.1.53}$$

$$\underset{\text{Spec}}{O}{}^{\text{BK}}\Phi_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}. \tag{2.1.54}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\widetilde{\Phi}^{r}_{\psi,\Gamma,A}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\widetilde{\Phi}^{I}_{\psi,\Gamma,A},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\breve{\Phi}^{r}_{\psi,\Gamma,A}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\breve{\Phi}^{I}_{\psi,\Gamma,A},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\Phi^{r}_{\psi,\Gamma,A}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\Phi^{I}_{\psi,\Gamma,A}.$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\widetilde{\Phi}^{r}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\widetilde{\Phi}^{I}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\breve{\Phi}^{r}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\breve{\Phi}^{I}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\Phi^{r}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\Phi^{I}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}.$$

Definition 2.1.10. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves}, \text{Perfectcomplex}, \text{Condensed}_{*} \tag{2.1.55}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}^{+}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.56}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}^{+}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}^{\dagger}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}^{\dagger}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.57}$$

$$\tag{2.1.58}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}^{+}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.59}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\nabla}^{+}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}^{\dagger}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\nabla}^{\dagger}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.60}$$

$$\tag{2.1.61}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta^{+}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.62}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla^{+}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta^{\dagger}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla^{\dagger}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}. \tag{2.1.63}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{r}_{\psi,\Gamma,A}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{I}_{\psi,\Gamma,A},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^{r}_{\psi,\Gamma,A}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^{I}_{\psi,\Gamma,A},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{r}_{\psi,\Gamma,A}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{I}_{\psi,\Gamma,A}.$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{r}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{I}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^{r}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \breve{\Phi}^{I}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{r}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{I}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}.$$

Proposition 2.1.11. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Perfectcomplex, Condensed}_{*} \tag{2.1.64}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}_{\psi,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.65}$$

$$\tag{2.1.66}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.67}$$

$$\tag{2.1.68}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.69}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,A}.$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \breve{\Phi}^I_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{2.1.70}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{2.1.71}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 2.1.12. Similar proposition holds for

$$\mathrm{Quasicoherentsheaves}, \mathrm{Perfectcomplex}, \mathrm{IndBanach}_*. \tag{2.1.72}$$

2.1.2 Frobenius Quasicoherent Modules II: Deformation in Banach Rings

Definition 2.1.13. Let ψ be a toric tower over $\mathbb{Q}_p$ as in [KL2, Chapter 7] with base $\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power:

$$\widetilde{\Delta}_{\psi,\Gamma},\widetilde{\nabla}_{\psi,\Gamma},\widetilde{\Phi}_{\psi,\Gamma},\widetilde{\Delta}^+_{\psi,\Gamma},\widetilde{\nabla}^+_{\psi,\Gamma},\widetilde{\Delta}^\dagger_{\psi,\Gamma},\widetilde{\nabla}^\dagger_{\psi,\Gamma},\widetilde{\Phi}^r_{\psi,\Gamma},\widetilde{\Phi}^I_{\psi,\Gamma},$$

$$\breve{\Delta}_{\psi,\Gamma},\breve{\nabla}_{\psi,\Gamma},\breve{\Phi}_{\psi,\Gamma},\breve{\Delta}^+_{\psi,\Gamma},\breve{\nabla}^+_{\psi,\Gamma},\breve{\Delta}^\dagger_{\psi,\Gamma},\breve{\nabla}^\dagger_{\psi,\Gamma},\breve{\Phi}^r_{\psi,\Gamma},\breve{\Phi}^I_{\psi,\Gamma},$$

$$\Delta_{\psi,\Gamma},\nabla_{\psi,\Gamma},\Phi_{\psi,\Gamma},\Delta^+_{\psi,\Gamma},\nabla^+_{\psi,\Gamma},\Delta^\dagger_{\psi,\Gamma},\nabla^\dagger_{\psi,\Gamma},\Phi^r_{\psi,\Gamma},\Phi^I_{\psi,\Gamma}.$$

We now consider the following rings with $-$ being any deforming Banach ring over $\mathbb{Q}_p$. Taking the product we have:

$$\widetilde{\Phi}_{\psi,\Gamma,-},\widetilde{\Phi}^r_{\psi,\Gamma,-},\widetilde{\Phi}^I_{\psi,\Gamma,-},$$

$$\breve{\Phi}_{\psi,\Gamma,-},\breve{\Phi}^r_{\psi,\Gamma,-},\breve{\Phi}^I_{\psi,\Gamma,-},$$

$$\Phi_{\psi,\Gamma,-},\Phi^r_{\psi,\Gamma,-},\Phi^I_{\psi,\Gamma,-}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma := \mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty,1)$-categories of $(\infty,1)$-modules.

Definition 2.1.14. First we consider the Bambozzi-Kremnizer spectrum $\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\Gamma,-},\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,-},\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,-},\tag{2.1.73}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}_{\psi,\Gamma,-},\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,-},\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,-},\tag{2.1.74}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi_{\psi,\Gamma,-},\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,-},\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,-}.\tag{2.1.75}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},\tag{2.1.76}$$

$$\tag{2.1.77}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},\tag{2.1.78}$$

$$\tag{2.1.79}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}. \tag{2.1.80}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^r_{\psi,\Gamma,-}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^I_{\psi,\Gamma,-}.$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^r_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^I_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^r_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^I_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 2.1.15. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Namely A will still as above as a Banach ring over $\mathbb{Q}_p$. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_{\psi,\Gamma}, \widetilde{\nabla}_{\psi,\Gamma}, \widetilde{\Phi}_{\psi,\Gamma}, \widetilde{\Delta}^+_{\psi,\Gamma}, \widetilde{\nabla}^+_{\psi,\Gamma}, \widetilde{\Delta}^\dagger_{\psi,\Gamma}, \widetilde{\nabla}^\dagger_{\psi,\Gamma}, \widetilde{\Phi}^r_{\psi,\Gamma}, \widetilde{\Phi}^I_{\psi,\Gamma},$$

$$\breve{\Delta}_{\psi,\Gamma}, \breve{\nabla}_{\psi,\Gamma}, \breve{\Phi}_{\psi,\Gamma}, \breve{\Delta}^+_{\psi,\Gamma}, \breve{\nabla}^+_{\psi,\Gamma}, \breve{\Delta}^\dagger_{\psi,\Gamma}, \breve{\nabla}^\dagger_{\psi,\Gamma}, \breve{\Phi}^r_{\psi,\Gamma}, \breve{\Phi}^I_{\psi,\Gamma},$$

$$\Delta_{\psi,\Gamma}, \nabla_{\psi,\Gamma}, \Phi_{\psi,\Gamma}, \Delta^+_{\psi,\Gamma}, \nabla^+_{\psi,\Gamma}, \Delta^\dagger_{\psi,\Gamma}, \nabla^\dagger_{\psi,\Gamma}, \Phi^r_{\psi,\Gamma}, \Phi^I_{\psi,\Gamma},$$

with A. Then we have the notations:

$$\widetilde{\Delta}_{\psi,\Gamma,-}, \widetilde{\nabla}_{\psi,\Gamma,-}, \widetilde{\Phi}_{\psi,\Gamma,-}, \widetilde{\Delta}^+_{\psi,\Gamma,-}, \widetilde{\nabla}^+_{\psi,\Gamma,-}, \widetilde{\Delta}^\dagger_{\psi,\Gamma,-}, \widetilde{\nabla}^\dagger_{\psi,\Gamma,-}, \widetilde{\Phi}^r_{\psi,\Gamma,-}, \widetilde{\Phi}^I_{\psi,\Gamma,-},$$

$$\breve{\Delta}_{\psi,\Gamma,-}, \breve{\nabla}_{\psi,\Gamma,-}, \breve{\Phi}_{\psi,\Gamma,-}, \breve{\Delta}^+_{\psi,\Gamma,-}, \breve{\nabla}^+_{\psi,\Gamma,-}, \breve{\Delta}^\dagger_{\psi,\Gamma,-}, \breve{\nabla}^\dagger_{\psi,\Gamma,-}, \breve{\Phi}^r_{\psi,\Gamma,-}, \breve{\Phi}^I_{\psi,\Gamma,-},$$

$$\Delta_{\psi,\Gamma,-}, \nabla_{\psi,\Gamma,-}, \Phi_{\psi,\Gamma,-}, \Delta^+_{\psi,\Gamma,-}, \nabla^+_{\psi,\Gamma,-}, \Delta^\dagger_{\psi,\Gamma,-}, \nabla^\dagger_{\psi,\Gamma,-}, \Phi^r_{\psi,\Gamma,-}, \Phi^I_{\psi,\Gamma,-}.$$

Definition 2.1.16. First we consider the Clausen-Scholze spectrum $\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,-}, \tag{2.1.81}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,-}, \tag{2.1.82}$$

$$\tag{2.1.83}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,-}, \breve{\nabla}_{\psi,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,-}, \tag{2.1.84}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,-}, \breve{\Phi}^I_{\psi,\Gamma,-}, \tag{2.1.85}$$

$$\tag{2.1.86}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Delta_{\psi,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\nabla_{\psi,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Phi_{\psi,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,-}, \tag{2.1.87}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,-}. \tag{2.1.88}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.89}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.90}$$

$$\tag{2.1.91}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.92}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.93}$$

$$\tag{2.1.94}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Delta_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\nabla_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Phi_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.95}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.1.96}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{\psi,\Gamma,-}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{\psi,\Gamma,-}.$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^r_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \breve{\Phi}^I_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}.$$

Definition 2.1.17. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves}, \text{IndBanach}_{*} \tag{2.1.97}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.98}$$

$$\tag{2.1.99}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.100}$$

$$\tag{2.1.101}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}. \tag{2.1.102}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^r_{\psi,\Gamma,-}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^I_{\psi,\Gamma,-}.$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^r_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^I_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^r_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^I_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}.$$

Definition 2.1.18. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Condensed}_{*} \tag{2.1.103}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{O}{}^{\text{CS}}\widetilde{\Delta}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{O}{}^{\text{CS}}\widetilde{\nabla}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{O}{}^{\text{CS}}\widetilde{\Phi}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{O}{}^{\text{CS}}\widetilde{\Delta}^{+}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.104}$$

$$\underset{\text{Spec}}{O}{}^{\text{CS}}\widetilde{\nabla}^{+}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{O}{}^{\text{CS}}\widetilde{\Delta}^{\dagger}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{O}{}^{\text{CS}}\widetilde{\nabla}^{\dagger}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.105}$$

$$\tag{2.1.106}$$

$$\underset{\text{Spec}}{O}{}^{\text{CS}}\breve{\Delta}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{O}{}^{\text{CS}}\breve{\Phi}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{O}{}^{\text{CS}}\breve{\Delta}^{+}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.107}$$

$$\underset{\text{Spec}}{O}{}^{\text{CS}}\breve{\nabla}^{+}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{O}{}^{\text{CS}}\breve{\Delta}^{\dagger}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{O}{}^{\text{CS}}\breve{\nabla}^{\dagger}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.108}$$

$$\tag{2.1.109}$$

$$\underset{\text{Spec}}{O}{}^{\text{CS}}\Delta_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{O}{}^{\text{CS}}\nabla_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{O}{}^{\text{CS}}\Phi_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{O}{}^{\text{CS}}\Delta^{+}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.110}$$

$$\underset{\text{Spec}}{O}{}^{\text{CS}}\nabla^{+}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{O}{}^{\text{CS}}\Delta^{\dagger}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{O}{}^{\text{CS}}\nabla^{\dagger}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}. \tag{2.1.111}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\text{homotopycolimit}_r \underset{\text{Spec}}{O}{}^{\text{CS}}\widetilde{\Phi}^{r}_{\psi,\Gamma,-}, \text{homotopylimit}_I \underset{\text{Spec}}{O}{}^{\text{CS}}\widetilde{\Phi}^{I}_{\psi,\Gamma,-},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{O}{}^{\text{CS}}\breve{\Phi}^{r}_{\psi,\Gamma,-}, \text{homotopylimit}_I \underset{\text{Spec}}{O}{}^{\text{CS}}\breve{\Phi}^{I}_{\psi,\Gamma,-},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{O}{}^{\text{CS}}\Phi^{r}_{\psi,\Gamma,-}, \text{homotopylimit}_I \underset{\text{Spec}}{O}{}^{\text{CS}}\Phi^{I}_{\psi,\Gamma,-}.$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{O}{}^{\text{CS}}\widetilde{\Phi}^{r}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \underset{\text{Spec}}{O}{}^{\text{CS}}\widetilde{\Phi}^{I}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{O}{}^{\text{CS}}\breve{\Phi}^{r}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \breve{\Phi}^{I}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{O}{}^{\text{CS}}\Phi^{r}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \underset{\text{Spec}}{O}{}^{\text{CS}}\Phi^{I}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}.$$

Proposition 2.1.19. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Condensed}_{*} \tag{2.1.112}$$

where $*$ is one of the following spaces:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.113}$$

$$\tag{2.1.114}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.115}$$

$$\tag{2.1.116}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.117}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,-}.$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \breve{\Phi}^I_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{2.1.118}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{2.1.119}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 2.1.20. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{IndBanach}_*. \tag{2.1.120}$$

Definition 2.1.21. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves}, \text{Perfectcomplex}, \text{IndBanach}_* \tag{2.1.121}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.122}$$

$$\tag{2.1.123}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.124}$$

$$\tag{2.1.125}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}. \tag{2.1.126}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^r_{\psi,\Gamma,-}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^I_{\psi,\Gamma,-}.$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^r_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^I_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^r_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^I_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}.$$

Definition 2.1.22. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves}, \text{Perfectcomplex}, \text{Condensed}_* \tag{2.1.127}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}^{+}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.128}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}^{+}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}^{\dagger}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}^{\dagger}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.129}$$

$$\tag{2.1.130}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}^{+}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.131}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\nabla}^{+}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}^{\dagger}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\nabla}^{\dagger}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.132}$$

$$\tag{2.1.133}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta^{+}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.134}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla^{+}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta^{\dagger}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla^{\dagger}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}. \tag{2.1.135}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{r}_{\psi,\Gamma,-}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{I}_{\psi,\Gamma,-},$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^{r}_{\psi,\Gamma,-}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^{I}_{\psi,\Gamma,-},$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{r}_{\psi,\Gamma,-}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{I}_{\psi,\Gamma,-}.$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{r}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{I}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^{r}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\, \breve{\Phi}^{I}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{r}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{I}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}.$$

Proposition 2.1.23. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Perfectcomplex, Condensed}_{*} \tag{2.1.136}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}_{\psi,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.137}$$

$$\tag{2.1.138}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.139}$$

$$\tag{2.1.140}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.141}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,-},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,-}.$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \breve{\Phi}^I_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{2.1.142}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{2.1.143}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 2.1.24. Similar proposition holds for

$$\mathrm{Quasicoherentsheaves}, \mathrm{Perfectcomplex}, \mathrm{IndBanach}_{*}. \tag{2.1.144}$$

2.1.3 Frobenius Quasicoherent Modules III: Deformation in $(\infty, 1)$-Ind-Banach Rings

Definition 2.1.25. Let ψ be a toric tower over $\mathbb{Q}_p$ as in [KL2, Chapter 7] with base $\mathbb{Q}_p \left\langle X_1^{\pm 1}, ..., X_k^{\pm 1} \right\rangle$. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power:

$$\widetilde{\Delta}_{\psi,\Gamma}, \widetilde{\nabla}_{\psi,\Gamma}, \widetilde{\Phi}_{\psi,\Gamma}, \widetilde{\Delta}^+_{\psi,\Gamma}, \widetilde{\nabla}^+_{\psi,\Gamma}, \widetilde{\Delta}^\dagger_{\psi,\Gamma}, \widetilde{\nabla}^\dagger_{\psi,\Gamma}, \widetilde{\Phi}^r_{\psi,\Gamma}, \widetilde{\Phi}^I_{\psi,\Gamma},$$

$$\breve{\Delta}_{\psi,\Gamma}, \breve{\nabla}_{\psi,\Gamma}, \breve{\Phi}_{\psi,\Gamma}, \breve{\Delta}^+_{\psi,\Gamma}, \breve{\nabla}^+_{\psi,\Gamma}, \breve{\Delta}^\dagger_{\psi,\Gamma}, \breve{\nabla}^\dagger_{\psi,\Gamma}, \breve{\Phi}^r_{\psi,\Gamma}, \breve{\Phi}^I_{\psi,\Gamma},$$

$$\Delta_{\psi,\Gamma}, \nabla_{\psi,\Gamma}, \Phi_{\psi,\Gamma}, \Delta^+_{\psi,\Gamma}, \nabla^+_{\psi,\Gamma}, \Delta^\dagger_{\psi,\Gamma}, \nabla^\dagger_{\psi,\Gamma}, \Phi^r_{\psi,\Gamma}, \Phi^I_{\psi,\Gamma}.$$

We now consider the following rings with $\square$ being a homotopy colimit

$$\underset{I}{\mathrm{homotopylimit}} \square_i \tag{2.1.145}$$

of $\mathbb{Q}_p \langle Y_1, ..., Y_i \rangle, i = 1, 2, ...$ in ∞-categories of simplicial ind-Banach rings in [BBBK]

$$\mathrm{SimplicialInd-BanachRings}_{\mathbb{Q}_p} \tag{2.1.146}$$

or animated analytic condensed commutative algebras in [CS2]

$$\mathrm{SimplicialAnalyticCondensed}_{\mathbb{Q}_p}. \tag{2.1.147}$$

Taking the product we have:

$$\widetilde{\Phi}_{\psi,\Gamma,\square}, \widetilde{\Phi}^r_{\psi,\Gamma,\square}, \widetilde{\Phi}^I_{\psi,\Gamma,\square},$$

$$\breve{\Phi}_{\psi,\Gamma,\square}, \breve{\Phi}^r_{\psi,\Gamma,\square}, \breve{\Phi}^I_{\psi,\Gamma,\square},$$

$$\Phi_{\psi,\Gamma,\square}, \Phi^r_{\psi,\Gamma,\square}, \Phi^I_{\psi,\Gamma,\square}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma := \mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty, 1)$-categories of $(\infty, 1)$-modules.

Definition 2.1.26. First we consider the Bambozzi-Kremnizer spectrum $\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\Gamma,\square}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,\square}, \tag{2.1.148}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}_{\psi,\Gamma,\square}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,\square}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,\square}, \tag{2.1.149}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi_{\psi,\Gamma,\square}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,\square}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,\square}. \tag{2.1.150}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.151}$$

$$\tag{2.1.152}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.153}$$

$$\tag{2.1.154}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.1.155}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}, \underset{I}{\mathrm{homotopylimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopycolimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,\square}, \underset{I}{\mathrm{homotopylimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopycolimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,\square}, \underset{I}{\mathrm{homotopylimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,\square}.$$

$$\underset{r}{\mathrm{homotopycolimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 2.1.27. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Namely A will still as above as a Banach ring over $\mathbb{Q}_p$. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_{\psi,\Gamma}, \widetilde{\nabla}_{\psi,\Gamma}, \widetilde{\Phi}_{\psi,\Gamma}, \widetilde{\Delta}^+_{\psi,\Gamma}, \widetilde{\nabla}^+_{\psi,\Gamma}, \widetilde{\Delta}^\dagger_{\psi,\Gamma}, \widetilde{\nabla}^\dagger_{\psi,\Gamma}, \widetilde{\Phi}^r_{\psi,\Gamma}, \widetilde{\Phi}^I_{\psi,\Gamma},$$

$$\breve{\Delta}_{\psi,\Gamma}, \breve{\nabla}_{\psi,\Gamma}, \breve{\Phi}_{\psi,\Gamma}, \breve{\Delta}^+_{\psi,\Gamma}, \breve{\nabla}^+_{\psi,\Gamma}, \breve{\Delta}^\dagger_{\psi,\Gamma}, \breve{\nabla}^\dagger_{\psi,\Gamma}, \breve{\Phi}^r_{\psi,\Gamma}, \breve{\Phi}^I_{\psi,\Gamma},$$

$$\Delta_{\psi,\Gamma}, \nabla_{\psi,\Gamma}, \Phi_{\psi,\Gamma}, \Delta^+_{\psi,\Gamma}, \nabla^+_{\psi,\Gamma}, \Delta^\dagger_{\psi,\Gamma}, \nabla^\dagger_{\psi,\Gamma}, \Phi^r_{\psi,\Gamma}, \Phi^I_{\psi,\Gamma},$$

with A. Then we have the notations:

$$\widetilde{\Delta}_{\psi,\Gamma,\square}, \widetilde{\nabla}_{\psi,\Gamma,\square}, \widetilde{\Phi}_{\psi,\Gamma,\square}, \widetilde{\Delta}^+_{\psi,\Gamma,\square}, \widetilde{\nabla}^+_{\psi,\Gamma,\square}, \widetilde{\Delta}^\dagger_{\psi,\Gamma,\square}, \widetilde{\nabla}^\dagger_{\psi,\Gamma,\square}, \widetilde{\Phi}^r_{\psi,\Gamma,\square}, \widetilde{\Phi}^I_{\psi,\Gamma,\square},$$

$$\breve{\Delta}_{\psi,\Gamma,\square}, \breve{\nabla}_{\psi,\Gamma,\square}, \breve{\Phi}_{\psi,\Gamma,\square}, \breve{\Delta}^+_{\psi,\Gamma,\square}, \breve{\nabla}^+_{\psi,\Gamma,\square}, \breve{\Delta}^\dagger_{\psi,\Gamma,\square}, \breve{\nabla}^\dagger_{\psi,\Gamma,\square}, \breve{\Phi}^r_{\psi,\Gamma,\square}, \breve{\Phi}^I_{\psi,\Gamma,\square},$$

$$\Delta_{\psi,\Gamma,\square}, \nabla_{\psi,\Gamma,\square}, \Phi_{\psi,\Gamma,\square}, \Delta^+_{\psi,\Gamma,\square}, \nabla^+_{\psi,\Gamma,\square}, \Delta^\dagger_{\psi,\Gamma,\square}, \nabla^\dagger_{\psi,\Gamma,\square}, \Phi^r_{\psi,\Gamma,\square}, \Phi^I_{\psi,\Gamma,\square}.$$

Definition 2.1.28. First we consider the Clausen-Scholze spectrum $\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,\square}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,\square}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,\square}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,\square}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,\square}, \tag{2.1.156}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,\square}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,\square}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\square}, \tag{2.1.157}$$

$$\tag{2.1.158}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,\square}, \breve{\nabla}_{\psi,\Gamma,\square}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,\square}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,\square}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,\square}, \tag{2.1.159}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,\square}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,\square}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\square}, \breve{\Phi}^I_{\psi,\Gamma,\square}, \tag{2.1.160}$$

$$\tag{2.1.161}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta_{\psi,\Gamma,\square}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla_{\psi,\Gamma,\square}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{\psi,\Gamma,\square}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,\square}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,\square}, \tag{2.1.162}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,\square}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,\square}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\square}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\square}. \tag{2.1.163}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.164}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.165}$$

$$\tag{2.1.166}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.167}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.168}$$

$$\tag{2.1.169}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.170}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.1.171}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^r_{\psi,\Gamma,\square}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{\psi,\Gamma,\square}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{\psi,\Gamma,\square}.$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^r_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}} \breve{\Phi}^I_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}.$$

Definition 2.1.29. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM][1]:

$$\text{Quasicoherentpresheaves}, \text{IndBanach}_{*} \tag{2.1.172}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.173}$$

$$\tag{2.1.174}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.175}$$

$$\tag{2.1.176}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}. \tag{2.1.177}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,\square},$$

[1]Here the categories are defined to be the corresponding homotopy colimits of the corresponding categories with respect to each $\square_I$.

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{BK}}\breve{\Phi}^r_{\psi,\Gamma,\square},\ \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{BK}}\breve{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{BK}}\Phi^r_{\psi,\Gamma,\square},\ \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{BK}}\Phi^I_{\psi,\Gamma,\square}.$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},\ \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{BK}}\breve{\Phi}^r_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},\ \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{BK}}\breve{\Phi}^I_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{BK}}\Phi^r_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},\ \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{BK}}\Phi^I_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}.$$

Definition 2.1.30. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Condensed}_* \tag{2.1.178}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{CS}}\widetilde{\Delta}_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{CS}}\widetilde{\nabla}_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{CS}}\widetilde{\Phi}_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.179}$$

$$\underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.180}$$

$$\tag{2.1.181}$$

$$\underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{CS}}\breve{\Delta}_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},\ \breve{\nabla}_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{CS}}\breve{\Phi}_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{CS}}\breve{\Delta}^+_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.182}$$

$$\underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{CS}}\breve{\nabla}^+_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.183}$$

$$\tag{2.1.184}$$

$$\underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{CS}}\Delta_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{CS}}\nabla_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{CS}}\Phi_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{CS}}\Delta^+_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.185}$$

$$\underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{CS}}\nabla^+_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{CS}}\Delta^\dagger_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{CS}}\nabla^\dagger_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}. \tag{2.1.186}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\square},\ \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{CS}}\breve{\Phi}^r_{\psi,\Gamma,\square},\ \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{CS}}\breve{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{CS}}\Phi^r_{\psi,\Gamma,\square},\ \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}\ {}^{\text{CS}}\Phi^I_{\psi,\Gamma,\square}.$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{homotopylimit}_I \breve{\Phi}^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 2.1.31. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Condensed}_* \tag{2.1.187}$$

where $*$ is one of the following spaces:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.188}$$

$$\tag{2.1.189}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.190}$$

$$\tag{2.1.191}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.192}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\square},$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\square}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,\square},$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\square}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\square}.$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{homotopylimit}_I \breve{\Phi}^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\, M_r, \tag{2.1.193}$$

$$\underset{I}{\text{homotopylimit}}\, M_I, \tag{2.1.194}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 2.1.32. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{IndBanach}_*. \tag{2.1.195}$$

Definition 2.1.33. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves}, \text{Perfectcomplex}, \text{IndBanach}_* \tag{2.1.196}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.197}$$

$$\tag{2.1.198}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.199}$$

$$\tag{2.1.200}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}. \tag{2.1.201}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}}\, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}, \underset{I}{\text{homotopylimit}}\, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\text{homotopycolimit}}\, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^r_{\psi,\Gamma,\square}, \underset{I}{\text{homotopylimit}}\, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\text{homotopycolimit}}\, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^r_{\psi,\Gamma,\square}, \underset{I}{\text{homotopylimit}}\, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^I_{\psi,\Gamma,\square}.$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^r_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^I_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^r_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^I_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}.$$

Definition 2.1.34. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Perfectcomplex, Condensed}_* \tag{2.1.202}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.203}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.204}$$

$$\tag{2.1.205}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}^+_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.206}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\nabla}^+_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.207}$$

$$\tag{2.1.208}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta^+_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.1.209}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla^+_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta^\dagger_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla^\dagger_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}. \tag{2.1.210}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^r_{\psi,\Gamma,\square}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{\psi,\Gamma,\square}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{\psi,\Gamma,\square}.$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^r_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}} \breve{\Phi}^I_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{\psi,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}.$$

Proposition 2.1.35. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves,Perfectcomplex,Condensed}_{*} \tag{2.1.211}$$

where $*$ is one of the following spaces:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.212}$$

$$\tag{2.1.213}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.214}$$

$$\tag{2.1.215}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.1.216}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}, \underset{I}{\mathrm{homotopylimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopycolimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\square}, \underset{I}{\mathrm{homotopylimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopycolimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\square}, \underset{I}{\mathrm{homotopylimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\square}.$$

$$\underset{r}{\mathrm{homotopycolimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\,\breve{\Phi}^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\, M_r, \tag{2.1.217}$$

$$\underset{I}{\mathrm{homotopylimit}}\, M_I, \tag{2.1.218}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 2.1.36. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{Perfectcomplex}, \text{IndBanach}_*. \tag{2.1.219}$$

2.2 Univariate Hodge Iwasawa Modules by Deformation

This chapter follows closely [T1], [T2], [T3], [T4], [T5], [T6], [KPX], [KP], [KL1], [KL2], [BK], [BBBK], [BBM], [KKM], [CS1], [CS2], [LBV].

2.2.1 Frobenius Quasicoherent Modules I

Definition 2.2.1. First we consider the Bambozzi-Kremnizer spectrum $\underset{\text{Spec}}{O}{}^{\text{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\underset{\text{Spec}}{O}{}^{\text{BK}}\widetilde{\Phi}_{\psi,A}, \underset{\text{Spec}}{O}{}^{\text{BK}}\widetilde{\Phi}^r_{\psi,A}, \underset{\text{Spec}}{O}{}^{\text{BK}}\widetilde{\Phi}^I_{\psi,A}, \tag{2.2.1}$$

$$\underset{\text{Spec}}{O}{}^{\text{BK}}\breve{\Phi}_{\psi,A}, \underset{\text{Spec}}{O}{}^{\text{BK}}\breve{\Phi}^r_{\psi,A}, \underset{\text{Spec}}{O}{}^{\text{BK}}\breve{\Phi}^I_{\psi,A}, \tag{2.2.2}$$

$$\underset{\text{Spec}}{O}{}^{\text{BK}}\Phi_{\psi,A}, \underset{\text{Spec}}{O}{}^{\text{BK}}\Phi^r_{\psi,A}, \underset{\text{Spec}}{O}{}^{\text{BK}}\Phi^I_{\psi,A}. \tag{2.2.3}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\underset{\text{Spec}}{O}{}^{\text{BK}}\widetilde{\Phi}_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.2.4}$$

$$\tag{2.2.5}$$

$$\underset{\text{Spec}}{O}{}^{\text{BK}}\breve{\Phi}_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.2.6}$$

$$\tag{2.2.7}$$

$$\underset{\text{Spec}}{O}{}^{\text{BK}}\Phi_{\psi,A}/\text{Fro}^{\mathbb{Z}}. \tag{2.2.8}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\widetilde{\Phi}^r_{\psi,A}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\widetilde{\Phi}^I_{\psi,A},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\breve{\Phi}^r_{\psi,A}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\breve{\Phi}^I_{\psi,A},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\Phi^r_{\psi,A}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\Phi^I_{\psi,A}.$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{\psi,A}/\text{Fro}^{\mathbb{Z}},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^r_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^I_{\psi,A}/\text{Fro}^{\mathbb{Z}},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^r_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^I_{\psi,A}/\text{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 2.2.2. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Namely A will still as above as a Banach ring over $\mathbb{Q}_p$. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_\psi, \widetilde{\nabla}_\psi, \widetilde{\Phi}_\psi, \widetilde{\Delta}^+_\psi, \widetilde{\nabla}^+_\psi, \widetilde{\Delta}^\dagger_\psi, \widetilde{\nabla}^\dagger_\psi, \widetilde{\Phi}^r_\psi, \widetilde{\Phi}^I_\psi,$$

$$\breve{\Delta}_\psi, \breve{\nabla}_\psi, \breve{\Phi}_\psi, \breve{\Delta}^+_\psi, \breve{\nabla}^+_\psi, \breve{\Delta}^\dagger_\psi, \breve{\nabla}^\dagger_\psi, \breve{\Phi}^r_\psi, \breve{\Phi}^I_\psi,$$

$$\Delta_\psi, \nabla_\psi, \Phi_\psi, \Delta^+_\psi, \nabla^+_\psi, \Delta^\dagger_\psi, \nabla^\dagger_\psi, \Phi^r_\psi, \Phi^I_\psi,$$

with A. Then we have the notations:

$$\widetilde{\Delta}_{\psi,A}, \widetilde{\nabla}_{\psi,A}, \widetilde{\Phi}_{\psi,A}, \widetilde{\Delta}^+_{\psi,A}, \widetilde{\nabla}^+_{\psi,A}, \widetilde{\Delta}^\dagger_{\psi,A}, \widetilde{\nabla}^\dagger_{\psi,A}, \widetilde{\Phi}^r_{\psi,A}, \widetilde{\Phi}^I_{\psi,A},$$

$$\breve{\Delta}_{\psi,A}, \breve{\nabla}_{\psi,A}, \breve{\Phi}_{\psi,A}, \breve{\Delta}^+_{\psi,A}, \breve{\nabla}^+_{\psi,A}, \breve{\Delta}^\dagger_{\psi,A}, \breve{\nabla}^\dagger_{\psi,A}, \breve{\Phi}^r_{\psi,A}, \breve{\Phi}^I_{\psi,A},$$

$$\Delta_{\psi,A}, \nabla_{\psi,A}, \Phi_{\psi,A}, \Delta^+_{\psi,A}, \nabla^+_{\psi,A}, \Delta^\dagger_{\psi,A}, \nabla^\dagger_{\psi,A}, \Phi^r_{\psi,A}, \Phi^I_{\psi,A}.$$

Definition 2.2.3. First we consider the Clausen-Scholze spectrum $\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}_{\psi,A}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}_{\psi,A}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}_{\psi,A}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}^+_{\psi,A}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}^+_{\psi,A}, \tag{2.2.9}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}^\dagger_{\psi,A}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}^\dagger_{\psi,A}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^r_{\psi,A}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^I_{\psi,A}, \tag{2.2.10}$$

$$\tag{2.2.11}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}_{\psi,A}, \breve{\nabla}_{\psi,A}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}_{\psi,A}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}^+_{\psi,A}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\nabla}^+_{\psi,A}, \tag{2.2.12}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}^\dagger_{\psi,A}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\nabla}^\dagger_{\psi,A}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^r_{\psi,A}, \breve{\Phi}^I_{\psi,A}, \tag{2.2.13}$$

$$\tag{2.2.14}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta_{\psi,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla_{\psi,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{\psi,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^+_{\psi,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^+_{\psi,A}, \tag{2.2.15}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^\dagger_{\psi,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^\dagger_{\psi,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{\psi,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{\psi,A}. \tag{2.2.16}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.17}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.18}$$

$$\tag{2.2.19}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.20}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.21}$$

$$\tag{2.2.22}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.23}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.2.24}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,A},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,A},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{\psi,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{\psi,A}.$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \breve{\Phi}^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 2.2.4. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\mathrm{Quasicoherentpresheaves}, \mathrm{IndBanach}_* \tag{2.2.25}$$

where $*$ is one of the following spaces:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.26}$$

$$\tag{2.2.27}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.28}$$

$$\tag{2.2.29}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.2.30}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,A}, \underset{I}{\mathrm{homotopylimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,A},$$

$$\underset{r}{\mathrm{homotopycolimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,A}, \underset{I}{\mathrm{homotopylimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,A},$$

$$\underset{r}{\mathrm{homotopycolimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^r_{\psi,A}, \underset{I}{\mathrm{homotopylimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^I_{\psi,A}.$$

$$\underset{r}{\mathrm{homotopycolimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 2.2.5. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\mathrm{Quasicoherentsheaves}, \mathrm{Condensed}_* \tag{2.2.31}$$

where $*$ is one of the following spaces:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.32}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.33}$$

$$\tag{2.2.34}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.35}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.36}$$

$$\tag{2.2.37}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^{+}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.38}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^{+}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^{\dagger}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^{\dagger}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.2.39}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,A}, \underset{I}{\mathrm{homotopylimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,A},$$

$$\underset{r}{\mathrm{homotopycolimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,A}, \underset{I}{\mathrm{homotopylimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,A},$$

$$\underset{r}{\mathrm{homotopycolimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{\psi,A}, \underset{I}{\mathrm{homotopylimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{\psi,A}.$$

$$\underset{r}{\mathrm{homotopycolimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\, \breve{\Phi}^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 2.2.6. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}_{,\mathrm{Condensed}_*} \tag{2.2.40}$$

where $*$ is one of the following spaces:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.41}$$

$$\tag{2.2.42}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.43}$$

$$\tag{2.2.44}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.45}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we

consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,A},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,A},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{\psi,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{\psi,A}.$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \breve{\Phi}^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{2.2.46}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{2.2.47}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 2.2.7. Similar proposition holds for

$$\mathrm{Quasicoherentsheaves}, \mathrm{IndBanach}_*. \tag{2.2.48}$$

Definition 2.2.8. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Perfectcomplex}, \mathrm{IndBanach}_* \tag{2.2.49}$$

where $*$ is one of the following spaces:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.50}$$

$$\tag{2.2.51}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.52}$$

$$\tag{2.2.53}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.2.54}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,A},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,A},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^r_{\psi,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^I_{\psi,A}.$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 2.2.9. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\mathrm{Quasicoherentsheaves, Perfectcomplex, Condensed}_* \tag{2.2.55}$$

where $*$ is one of the following spaces:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.56}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.57}$$

$$\tag{2.2.58}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.59}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.60}$$

$$\tag{2.2.61}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.62}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^+_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^\dagger_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.2.63}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,A}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,A},$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,A}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,A},$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{\psi,A}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{\psi,A}.$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{homotopylimit}_I \breve{\Phi}^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 2.2.10. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Perfectcomplex}, \mathrm{Condensed}_* \tag{2.2.64}$$

where $*$ is one of the following spaces:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.65}$$

$$\tag{2.2.66}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.67}$$

$$\tag{2.2.68}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{\psi,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.69}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,A}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,A},$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,A}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,A},$$

$$\underset{r}{\text{homotopycolimit}}\ \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\Phi^r_{\psi,A}, \underset{I}{\text{homotopylimit}}\ \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\Phi^I_{\psi,A}.$$

$$\underset{r}{\text{homotopycolimit}}\ \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\widetilde{\Phi}^r_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\widetilde{\Phi}^I_{\psi,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\breve{\Phi}^r_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \breve{\Phi}^I_{\psi,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\Phi^r_{\psi,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \mathcal{O}_{\text{Spec}}{}^{\text{CS}}\Phi^I_{\psi,A}/\text{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{2.2.70}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{2.2.71}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 2.2.11. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{Perfectcomplex}, \text{IndBanach}_*. \tag{2.2.72}$$

2.2.2 Frobenius Quasicoherent Modules II: Deformation in Banach Rings

Definition 2.2.12. Let ψ be a toric tower over $\mathbb{Q}_p$ as in [KL2, Chapter 7] with base $\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power[2]:

$$\widetilde{\Delta}_\psi, \widetilde{\nabla}_\psi, \widetilde{\Phi}_\psi, \widetilde{\Delta}^+_\psi, \widetilde{\nabla}^+_\psi, \widetilde{\Delta}^\dagger_\psi, \widetilde{\nabla}^\dagger_\psi, \widetilde{\Phi}^r_\psi, \widetilde{\Phi}^I_\psi,$$

$$\breve{\Delta}_\psi, \breve{\nabla}_\psi, \breve{\Phi}_\psi, \breve{\Delta}^+_\psi, \breve{\nabla}^+_\psi, \breve{\Delta}^\dagger_\psi, \breve{\nabla}^\dagger_\psi, \breve{\Phi}^r_\psi, \breve{\Phi}^I_\psi,$$

$$\Delta_\psi, \nabla_\psi, \Phi_\psi, \Delta^+_\psi, \nabla^+_\psi, \Delta^\dagger_\psi, \nabla^\dagger_\psi, \Phi^r_\psi, \Phi^I_\psi.$$

We now consider the following rings with $-$ being any deforming Banach ring over $\mathbb{Q}_p$. Taking the product we have:

$$\widetilde{\Phi}_{\psi,-}, \widetilde{\Phi}^r_{\psi,-}, \widetilde{\Phi}^I_{\psi,-},$$

$$\breve{\Phi}_{\psi,-}, \breve{\Phi}^r_{\psi,-}, \breve{\Phi}^I_{\psi,-},$$

$$\Phi_{\psi,-}, \Phi^r_{\psi,-}, \Phi^I_{\psi,-}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma := \mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty,1)$-categories of $(\infty,1)$-modules.

Definition 2.2.13. First we consider the Bambozzi-Kremnizer spectrum $\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,-}, \tag{2.2.73}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}_{\psi,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,-}, \tag{2.2.74}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi_{\psi,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^r_{\psi,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^I_{\psi,-}. \tag{2.2.75}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.76}$$

$$\tag{2.2.77}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.78}$$

$$\tag{2.2.79}$$

[2]Here $|\Gamma| = 1$.

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.2.80}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,-},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,-},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^r_{\psi,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^I_{\psi,-}.$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^r_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^I_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 2.2.14. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Namely A will still as above as a Banach ring over $\mathbb{Q}_p$. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_\psi, \widetilde{\nabla}_\psi, \widetilde{\Phi}_\psi, \widetilde{\Delta}^+_\psi, \widetilde{\nabla}^+_\psi, \widetilde{\Delta}^\dagger_\psi, \widetilde{\nabla}^\dagger_\psi, \widetilde{\Phi}^r_\psi, \widetilde{\Phi}^I_\psi,$$

$$\breve{\Delta}_\psi, \breve{\nabla}_\psi, \breve{\Phi}_\psi, \breve{\Delta}^+_\psi, \breve{\nabla}^+_\psi, \breve{\Delta}^\dagger_\psi, \breve{\nabla}^\dagger_\psi, \breve{\Phi}^r_\psi, \breve{\Phi}^I_\psi,$$

$$\Delta_\psi, \nabla_\psi, \Phi_\psi, \Delta^+_\psi, \nabla^+_\psi, \Delta^\dagger_\psi, \nabla^\dagger_\psi, \Phi^r_\psi, \Phi^I_\psi,$$

with A. Then we have the notations:

$$\widetilde{\Delta}_{\psi,-}, \widetilde{\nabla}_{\psi,-}, \widetilde{\Phi}_{\psi,-}, \widetilde{\Delta}^+_{\psi,-}, \widetilde{\nabla}^+_{\psi,-}, \widetilde{\Delta}^\dagger_{\psi,-}, \widetilde{\nabla}^\dagger_{\psi,-}, \widetilde{\Phi}^r_{\psi,-}, \widetilde{\Phi}^I_{\psi,-},$$

$$\breve{\Delta}_{\psi,-}, \breve{\nabla}_{\psi,-}, \breve{\Phi}_{\psi,-}, \breve{\Delta}^+_{\psi,-}, \breve{\nabla}^+_{\psi,-}, \breve{\Delta}^\dagger_{\psi,-}, \breve{\nabla}^\dagger_{\psi,-}, \breve{\Phi}^r_{\psi,-}, \breve{\Phi}^I_{\psi,-},$$

$$\Delta_{\psi,-}, \nabla_{\psi,-}, \Phi_{\psi,-}, \Delta^+_{\psi,-}, \nabla^+_{\psi,-}, \Delta^\dagger_{\psi,-}, \nabla^\dagger_{\psi,-}, \Phi^r_{\psi,-}, \Phi^I_{\psi,-}.$$

Definition 2.2.15. First we consider the Clausen-Scholze spectrum $\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^{+}_{\psi,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^{+}_{\psi,-}, \tag{2.2.81}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^{\dagger}_{\psi,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^{\dagger}_{\psi,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{\psi,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{\psi,-}, \tag{2.2.82}$$

$$\tag{2.2.83}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}_{\psi,-}, \breve{\nabla}_{\psi,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{\psi,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^{+}_{\psi,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^{+}_{\psi,-}, \tag{2.2.84}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^{\dagger}_{\psi,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^{\dagger}_{\psi,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^{r}_{\psi,-}, \breve{\Phi}^{I}_{\psi,-}, \tag{2.2.85}$$

$$\tag{2.2.86}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta_{\psi,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla_{\psi,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{\psi,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^{+}_{\psi,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^{+}_{\psi,-}, \tag{2.2.87}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^{\dagger}_{\psi,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^{\dagger}_{\psi,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{r}_{\psi,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{I}_{\psi,-}. \tag{2.2.88}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^{+}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.89}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^{+}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^{\dagger}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^{\dagger}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.90}$$

$$\tag{2.2.91}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^{+}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.92}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^{+}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^{\dagger}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^{\dagger}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.93}$$

$$\tag{2.2.94}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^{+}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.95}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^{+}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^{\dagger}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^{\dagger}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.2.96}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{\psi,-}, \underset{I}{\mathrm{homotopylimit}}\, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{\psi,-},$$

$$\underset{r}{\mathrm{homotopycolimit}}\, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^{r}_{\psi,-}, \underset{I}{\mathrm{homotopylimit}}\, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^{I}_{\psi,-},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{\psi,-}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{\psi,-}.$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^r_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^I_{\psi,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^r_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \breve{\Phi}^I_{\psi,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{\psi,-}/\text{Fro}^{\mathbb{Z}}.$$

Definition 2.2.16. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves, IndBanach}_* \tag{2.2.97}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.2.98}$$

$$\tag{2.2.99}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.2.100}$$

$$\tag{2.2.101}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi_{\psi,-}/\text{Fro}^{\mathbb{Z}}. \tag{2.2.102}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{\psi,-}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{\psi,-},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^r_{\psi,-}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^I_{\psi,-},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^r_{\psi,-}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^I_{\psi,-}.$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{\psi,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^r_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^I_{\psi,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^r_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^I_{\psi,-}/\text{Fro}^{\mathbb{Z}}.$$

Definition 2.2.17. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\mathrm{Quasicoherentsheaves, Condensed}_{*} \tag{2.2.103}$$

where $*$ is one of the following spaces:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^{+}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.104}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^{+}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^{\dagger}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^{\dagger}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.105}$$

$$\tag{2.2.106}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^{+}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.107}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^{+}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^{\dagger}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^{\dagger}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.108}$$

$$\tag{2.2.109}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^{+}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.110}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^{+}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^{\dagger}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^{\dagger}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.2.111}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{\psi,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{\psi,-},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^{r}_{\psi,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^{I}_{\psi,-},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{r}_{\psi,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{I}_{\psi,-}.$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^{r}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \breve{\Phi}^{I}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{r}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{I}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 2.2.18. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves, Condensed}_{*} \tag{2.2.112}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{O}{}^{\text{CS}}\widetilde{\Phi}_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.2.113}$$

$$\tag{2.2.114}$$

$$\underset{\text{Spec}}{O}{}^{\text{CS}}\breve{\Phi}_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.2.115}$$

$$\tag{2.2.116}$$

$$\underset{\text{Spec}}{O}{}^{\text{CS}}\Phi_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.2.117}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{CS}}\widetilde{\Phi}^r_{\psi,-}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{O}{}^{\text{CS}}\widetilde{\Phi}^I_{\psi,-},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{CS}}\breve{\Phi}^r_{\psi,-}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{O}{}^{\text{CS}}\breve{\Phi}^I_{\psi,-},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{CS}}\Phi^r_{\psi,-}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{O}{}^{\text{CS}}\Phi^I_{\psi,-}.$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{CS}}\widetilde{\Phi}^r_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{O}{}^{\text{CS}}\widetilde{\Phi}^I_{\psi,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{CS}}\breve{\Phi}^r_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \breve{\Phi}^I_{\psi,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{CS}}\Phi^r_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{O}{}^{\text{CS}}\Phi^I_{\psi,-}/\text{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{2.2.118}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{2.2.119}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 2.2.19. Similar proposition holds for

$$\text{Quasicoherentsheaves, IndBanach}_*. \tag{2.2.120}$$

Definition 2.2.20. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves, Perfectcomplex, IndBanach}_* \tag{2.2.121}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{O}{}^{\text{BK}}\widetilde{\Phi}_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.2.122}$$

$$\tag{2.2.123}$$

$$\underset{\text{Spec}}{O}{}^{\text{BK}}\breve{\Phi}_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.2.124}$$

$$\tag{2.2.125}$$

$$\underset{\text{Spec}}{O}{}^{\text{BK}}\Phi_{\psi,-}/\text{Fro}^{\mathbb{Z}}. \tag{2.2.126}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\widetilde{\Phi}^r_{\psi,-}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\widetilde{\Phi}^I_{\psi,-},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\breve{\Phi}^r_{\psi,-}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\breve{\Phi}^I_{\psi,-},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\Phi^r_{\psi,-}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\Phi^I_{\psi,-}.$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\widetilde{\Phi}^r_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\widetilde{\Phi}^I_{\psi,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\breve{\Phi}^r_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\breve{\Phi}^I_{\psi,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\Phi^r_{\psi,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{O}{}^{\text{BK}}\Phi^I_{\psi,-}/\text{Fro}^{\mathbb{Z}}.$$

Definition 2.2.21. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Perfectcomplex, Condensed}_* \tag{2.2.127}$$

where $*$ is one of the following spaces:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^{+}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.128}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^{+}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^{\dagger}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^{\dagger}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.129}$$

$$\tag{2.2.130}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^{+}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.131}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^{+}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^{\dagger}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^{\dagger}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.132}$$

$$\tag{2.2.133}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^{+}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.134}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^{+}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^{\dagger}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^{\dagger}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.2.135}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{\psi,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{\psi,-},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^{r}_{\psi,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^{I}_{\psi,-},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{r}_{\psi,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{I}_{\psi,-}.$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^{r}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \breve{\Phi}^{I}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{r}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{I}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 2.2.22. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Perfectcomplex}, \mathrm{Condensed}_{*} \tag{2.2.136}$$

where $*$ is one of the following spaces:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.137}$$

$$\tag{2.2.138}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.139}$$

$$\tag{2.2.140}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.2.141}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{\psi,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{\psi,-},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^{r}_{\psi,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^{I}_{\psi,-},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{r}_{\psi,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{I}_{\psi,-}.$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^{r}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \breve{\Phi}^{I}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{r}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{I}_{\psi,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{2.2.142}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{2.2.143}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 2.2.23. Similar proposition holds for

$$\mathrm{Quasicoherentsheaves}, \mathrm{Perfectcomplex}, \mathrm{IndBanach}_*. \tag{2.2.144}$$

2.3 Multivariate Hodge Iwasawa Prestacks by Deformation

This chapter follows closely [T1], [T2], [T3], [T4], [T5], [T6], [KPX], [KP], [KL1], [KL2], [BK], [BBBK], [BBM], [KKM], [CS1], [CS2], [CKZ], [PZ], [BCM], [LBV].

2.3.1 Frobenius Quasicoherent Prestacks I

Definition 2.3.1. First we consider the Bambozzi-Kremnizer spectrum $\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}_{*,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,A}, \tag{2.3.1}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}_{*,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,A}, \tag{2.3.2}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi_{*,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^r_{*,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^I_{*,\Gamma,A}. \tag{2.3.3}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.4}$$

$$\tag{2.3.5}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.6}$$

$$\tag{2.3.7}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.3.8}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^r_{*,\Gamma,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^I_{*,\Gamma,A}.$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^r_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^I_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 2.3.2. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Namely A will still as above as a Banach ring over $\mathbb{Q}_p$. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_{*,\Gamma}, \widetilde{\nabla}_{*,\Gamma}, \widetilde{\Phi}_{*,\Gamma}, \widetilde{\Delta}^+_{*,\Gamma}, \widetilde{\nabla}^+_{*,\Gamma}, \widetilde{\Delta}^\dagger_{*,\Gamma}, \widetilde{\nabla}^\dagger_{*,\Gamma}, \widetilde{\Phi}^r_{*,\Gamma}, \widetilde{\Phi}^I_{*,\Gamma},$$

$$\breve{\Delta}_{*,\Gamma}, \breve{\nabla}_{*,\Gamma}, \breve{\Phi}_{*,\Gamma}, \breve{\Delta}^+_{*,\Gamma}, \breve{\nabla}^+_{*,\Gamma}, \breve{\Delta}^\dagger_{*,\Gamma}, \breve{\nabla}^\dagger_{*,\Gamma}, \breve{\Phi}^r_{*,\Gamma}, \breve{\Phi}^I_{*,\Gamma},$$

$$\Delta_{*,\Gamma}, \nabla_{*,\Gamma}, \Phi_{*,\Gamma}, \Delta^+_{*,\Gamma}, \nabla^+_{*,\Gamma}, \Delta^\dagger_{*,\Gamma}, \nabla^\dagger_{*,\Gamma}, \Phi^r_{*,\Gamma}, \Phi^I_{*,\Gamma},$$

with A. Then we have the notations:

$$\widetilde{\Delta}_{*,\Gamma,A}, \widetilde{\nabla}_{*,\Gamma,A}, \widetilde{\Phi}_{*,\Gamma,A}, \widetilde{\Delta}^+_{*,\Gamma,A}, \widetilde{\nabla}^+_{*,\Gamma,A}, \widetilde{\Delta}^\dagger_{*,\Gamma,A}, \widetilde{\nabla}^\dagger_{*,\Gamma,A}, \widetilde{\Phi}^r_{*,\Gamma,A}, \widetilde{\Phi}^I_{*,\Gamma,A},$$

$$\breve{\Delta}_{*,\Gamma,A}, \breve{\nabla}_{*,\Gamma,A}, \breve{\Phi}_{*,\Gamma,A}, \breve{\Delta}^+_{*,\Gamma,A}, \breve{\nabla}^+_{*,\Gamma,A}, \breve{\Delta}^\dagger_{*,\Gamma,A}, \breve{\nabla}^\dagger_{*,\Gamma,A}, \breve{\Phi}^r_{*,\Gamma,A}, \breve{\Phi}^I_{*,\Gamma,A},$$

$$\Delta_{*,\Gamma,A}, \nabla_{*,\Gamma,A}, \Phi_{*,\Gamma,A}, \Delta^+_{*,\Gamma,A}, \nabla^+_{*,\Gamma,A}, \Delta^\dagger_{*,\Gamma,A}, \nabla^\dagger_{*,\Gamma,A}, \Phi^r_{*,\Gamma,A}, \Phi^I_{*,\Gamma,A}.$$

Definition 2.3.3. First we consider the Clausen-Scholze spectrum $\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Delta}_{*,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\nabla}_{*,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,\Gamma,A}, \tag{2.3.9}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,A}, \tag{2.3.10}$$

(2.3.11)

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,A}, \breve{\nabla}_{*,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Delta}^+_{*,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\nabla}^+_{*,\Gamma,A}, \tag{2.3.12}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,A}, \breve{\Phi}^I_{*,\Gamma,A}, \tag{2.3.13}$$

(2.3.14)

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Delta_{*,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\nabla_{*,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Phi_{*,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Delta^+_{*,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\nabla^+_{*,\Gamma,A}, \tag{2.3.15}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Delta^\dagger_{*,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\nabla^\dagger_{*,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Phi^r_{*,\Gamma,A}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Phi^I_{*,\Gamma,A}. \tag{2.3.16}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Delta}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\nabla}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.17}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.18}$$

(2.3.19)

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^{+}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.20}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^{+}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^{\dagger}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^{\dagger}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.21}$$

$$\tag{2.3.22}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^{+}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.23}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^{+}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^{\dagger}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^{\dagger}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.3.24}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,\Gamma,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,\Gamma,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^{I}_{*,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{r}_{*,\Gamma,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{I}_{*,\Gamma,A}.$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \breve{\Phi}^{I}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{r}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{I}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 2.3.4. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\mathrm{Quasicoherentpresheaves}, \mathrm{IndBanach}_{*} \tag{2.3.25}$$

where $*$ is one of the following spaces:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.26}$$

$$\tag{2.3.27}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.28}$$

$$\tag{2.3.29}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.3.30}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{*,\Gamma,A}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{*,\Gamma,A},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^r_{*,\Gamma,A}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^I_{*,\Gamma,A},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^r_{*,\Gamma,A}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^I_{*,\Gamma,A}.$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^r_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^I_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^r_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^I_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}.$$

Definition 2.3.5. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Condensed}_* \tag{2.3.31}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}^+_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.32}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}^+_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}^\dagger_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}^\dagger_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.33}$$

$$\tag{2.3.34}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}^+_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.35}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\nabla}^+_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}^\dagger_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\nabla}^\dagger_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.36}$$

$$\tag{2.3.37}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta^+_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.38}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla^+_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta^\dagger_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla^\dagger_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}. \tag{2.3.39}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^r_{*,\Gamma,A}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^I_{*,\Gamma,A},$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,A}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^I_{*,\Gamma,A},$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{*,\Gamma,A}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{*,\Gamma,A}.$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{homotopylimit}_I \breve{\Phi}^I_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 2.3.6. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Condensed}_* \tag{2.3.40}$$

where $*$ is one of the following spaces:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.41}$$

$$\tag{2.3.42}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.43}$$

$$\tag{2.3.44}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.45}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,A}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,A},$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,A}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^I_{*,\Gamma,A},$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{*,\Gamma,A}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{*,\Gamma,A}.$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^r_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \breve{\Phi}^I_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{2.3.46}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{2.3.47}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 2.3.7. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{IndBanach}_*. \tag{2.3.48}$$

Definition 2.3.8. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves}, \text{Perfectcomplex}, \text{IndBanach}_* \tag{2.3.49}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.50}$$

$$\tag{2.3.51}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.52}$$

$$\tag{2.3.53}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}. \tag{2.3.54}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{*,\Gamma,A}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{*,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopycolimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,A}, \underset{I}{\mathrm{homotopylimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopycolimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^r_{*,\Gamma,A}, \underset{I}{\mathrm{homotopylimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^I_{*,\Gamma,A}.$$

$$\underset{r}{\mathrm{homotopycolimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^r_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^I_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 2.3.9. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\mathrm{Quasicoherentsheaves}, \mathrm{Perfectcomplex}, \mathrm{Condensed}_* \tag{2.3.55}$$

where $*$ is one of the following spaces:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.56}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.57}$$

$$\tag{2.3.58}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^+_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.59}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^+_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.60}$$

$$\tag{2.3.61}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^+_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.62}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^+_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^\dagger_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^\dagger_{*,\Gamma,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.3.63}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,A}, \underset{I}{\mathrm{homotopylimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopycolimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,A}, \underset{I}{\mathrm{homotopylimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^I_{*,\Gamma,A},$$

$$\underset{r}{\mathrm{homotopycolimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{*,\Gamma,A}, \underset{I}{\mathrm{homotopylimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{*,\Gamma,A}.$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{CS}}\widetilde{\Phi}^r_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{O}{}^{\text{CS}}\widetilde{\Phi}^I_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{CS}}\breve{\Phi}^r_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \breve{\Phi}^I_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{CS}}\Phi^r_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{O}{}^{\text{CS}}\Phi^I_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}.$$

Proposition 2.3.10. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Perfectcomplex, Condensed}_* \tag{2.3.64}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{O}{}^{\text{CS}}\widetilde{\Phi}_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.65}$$

$$\tag{2.3.66}$$

$$\underset{\text{Spec}}{O}{}^{\text{CS}}\breve{\Phi}_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.67}$$

$$\tag{2.3.68}$$

$$\underset{\text{Spec}}{O}{}^{\text{CS}}\Phi_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.69}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{CS}}\widetilde{\Phi}^r_{*,\Gamma,A}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{O}{}^{\text{CS}}\widetilde{\Phi}^I_{*,\Gamma,A},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{CS}}\breve{\Phi}^r_{*,\Gamma,A}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{O}{}^{\text{CS}}\breve{\Phi}^I_{*,\Gamma,A},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{CS}}\Phi^r_{*,\Gamma,A}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{O}{}^{\text{CS}}\Phi^I_{*,\Gamma,A}.$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{CS}}\widetilde{\Phi}^r_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{O}{}^{\text{CS}}\widetilde{\Phi}^I_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{CS}}\breve{\Phi}^r_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \breve{\Phi}^I_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{O}{}^{\text{CS}}\Phi^r_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{O}{}^{\text{CS}}\Phi^I_{*,\Gamma,A}/\text{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\mathrm{homotopycolimit}_r M_r, \tag{2.3.70}$$

$$\mathrm{homotopylimit}_I M_I, \tag{2.3.71}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 2.3.11. Similar proposition holds for

$$\mathrm{Quasicoherentsheaves}, \mathrm{Perfectcomplex}, \mathrm{IndBanach}_*. \tag{2.3.72}$$

2.3.2 Frobenius Quasicoherent Prestacks II: Deformation in Banach Rings

Definition 2.3.12. We now consider the pro-étale site of $\mathrm{Spa}\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$, denote that by $*$. To be more accurate we replace one component for Γ with the pro-étale site of $\mathrm{Spa}\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$. And we treat then all the functor to be prestacks for this site. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power:

$$\widetilde{\Delta}_{*,\Gamma},\widetilde{\nabla}_{*,\Gamma},\widetilde{\Phi}_{*,\Gamma},\widetilde{\Delta}^+_{*,\Gamma},\widetilde{\nabla}^+_{*,\Gamma},\widetilde{\Delta}^\dagger_{*,\Gamma},\widetilde{\nabla}^\dagger_{*,\Gamma},\widetilde{\Phi}^r_{*,\Gamma},\widetilde{\Phi}^I_{*,\Gamma},$$

$$\breve{\Delta}_{*,\Gamma},\breve{\nabla}_{*,\Gamma},\breve{\Phi}_{*,\Gamma},\breve{\Delta}^+_{*,\Gamma},\breve{\nabla}^+_{*,\Gamma},\breve{\Delta}^\dagger_{*,\Gamma},\breve{\nabla}^\dagger_{*,\Gamma},\breve{\Phi}^r_{*,\Gamma},\breve{\Phi}^I_{*,\Gamma},$$

$$\Delta_{*,\Gamma},\nabla_{*,\Gamma},\Phi_{*,\Gamma},\Delta^+_{*,\Gamma},\nabla^+_{*,\Gamma},\Delta^\dagger_{*,\Gamma},\nabla^\dagger_{*,\Gamma},\Phi^r_{*,\Gamma},\Phi^I_{*,\Gamma}.$$

We now consider the following rings with $-$ being any deforming Banach ring over $\mathbb{Q}_p$. Taking the product we have:

$$\widetilde{\Phi}_{*,\Gamma,-},\widetilde{\Phi}^r_{*,\Gamma,-},\widetilde{\Phi}^I_{*,\Gamma,-},$$

$$\breve{\Phi}_{*,\Gamma,-},\breve{\Phi}^r_{*,\Gamma,-},\breve{\Phi}^I_{*,\Gamma,-},$$

$$\Phi_{*,\Gamma,-},\Phi^r_{*,\Gamma,-},\Phi^I_{*,\Gamma,-}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma:=\mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty,1)$-categories of $(\infty,1)$-modules.

Definition 2.3.13. First we consider the Bambozzi-Kremnizer spectrum $\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}_{*,\Gamma,-},\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,-},\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,-},\tag{2.3.73}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}_{*,\Gamma,-},\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,-},\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,-},\tag{2.3.74}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi_{*,\Gamma,-},\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^r_{*,\Gamma,-},\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^I_{*,\Gamma,-}.\tag{2.3.75}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},\tag{2.3.76}$$

$$\tag{2.3.77}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},\tag{2.3.78}$$

$$\tag{2.3.79}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}. \tag{2.3.80}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{*,\Gamma,-}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{*,\Gamma,-},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^r_{*,\Gamma,-}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^I_{*,\Gamma,-},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^r_{*,\Gamma,-}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^I_{*,\Gamma,-}.$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^r_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^I_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^r_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^I_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 2.3.14. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Namely A will still as above as a Banach ring over $\mathbb{Q}_p$. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_{*,\Gamma}, \widetilde{\nabla}_{*,\Gamma}, \widetilde{\Phi}_{*,\Gamma}, \widetilde{\Delta}^+_{*,\Gamma}, \widetilde{\nabla}^+_{*,\Gamma}, \widetilde{\Delta}^\dagger_{*,\Gamma}, \widetilde{\nabla}^\dagger_{*,\Gamma}, \widetilde{\Phi}^r_{*,\Gamma}, \widetilde{\Phi}^I_{*,\Gamma},$$

$$\breve{\Delta}_{*,\Gamma}, \breve{\nabla}_{*,\Gamma}, \breve{\Phi}_{*,\Gamma}, \breve{\Delta}^+_{*,\Gamma}, \breve{\nabla}^+_{*,\Gamma}, \breve{\Delta}^\dagger_{*,\Gamma}, \breve{\nabla}^\dagger_{*,\Gamma}, \breve{\Phi}^r_{*,\Gamma}, \breve{\Phi}^I_{*,\Gamma},$$

$$\Delta_{*,\Gamma}, \nabla_{*,\Gamma}, \Phi_{*,\Gamma}, \Delta^+_{*,\Gamma}, \nabla^+_{*,\Gamma}, \Delta^\dagger_{*,\Gamma}, \nabla^\dagger_{*,\Gamma}, \Phi^r_{*,\Gamma}, \Phi^I_{*,\Gamma},$$

with A. Then we have the notations:

$$\widetilde{\Delta}_{*,\Gamma,-}, \widetilde{\nabla}_{*,\Gamma,-}, \widetilde{\Phi}_{*,\Gamma,-}, \widetilde{\Delta}^+_{*,\Gamma,-}, \widetilde{\nabla}^+_{*,\Gamma,-}, \widetilde{\Delta}^\dagger_{*,\Gamma,-}, \widetilde{\nabla}^\dagger_{*,\Gamma,-}, \widetilde{\Phi}^r_{*,\Gamma,-}, \widetilde{\Phi}^I_{*,\Gamma,-},$$

$$\breve{\Delta}_{*,\Gamma,-}, \breve{\nabla}_{*,\Gamma,-}, \breve{\Phi}_{*,\Gamma,-}, \breve{\Delta}^+_{*,\Gamma,-}, \breve{\nabla}^+_{*,\Gamma,-}, \breve{\Delta}^\dagger_{*,\Gamma,-}, \breve{\nabla}^\dagger_{*,\Gamma,-}, \breve{\Phi}^r_{*,\Gamma,-}, \breve{\Phi}^I_{*,\Gamma,-},$$

$$\Delta_{*,\Gamma,-}, \nabla_{*,\Gamma,-}, \Phi_{*,\Gamma,-}, \Delta^+_{*,\Gamma,-}, \nabla^+_{*,\Gamma,-}, \Delta^\dagger_{*,\Gamma,-}, \nabla^\dagger_{*,\Gamma,-}, \Phi^r_{*,\Gamma,-}, \Phi^I_{*,\Gamma,-}.$$

Definition 2.3.15. First we consider the Clausen-Scholze spectrum $\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Delta}_{*,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\nabla}_{*,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Delta}^{+}_{*,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\nabla}^{+}_{*,\Gamma,-}, \tag{2.3.81}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Delta}^{\dagger}_{*,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\nabla}^{\dagger}_{*,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,\Gamma,-}, \tag{2.3.82}$$

$$\tag{2.3.83}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,-}, \breve{\nabla}_{*,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Delta}^{+}_{*,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\nabla}^{+}_{*,\Gamma,-}, \tag{2.3.84}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Delta}^{\dagger}_{*,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\nabla}^{\dagger}_{*,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,\Gamma,-}, \breve{\Phi}^{I}_{*,\Gamma,-}, \tag{2.3.85}$$

$$\tag{2.3.86}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Delta_{*,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\nabla_{*,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Phi_{*,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Delta^{+}_{*,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\nabla^{+}_{*,\Gamma,-}, \tag{2.3.87}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Delta^{\dagger}_{*,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\nabla^{\dagger}_{*,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Phi^{r}_{*,\Gamma,-}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Phi^{I}_{*,\Gamma,-}. \tag{2.3.88}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Delta}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\nabla}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Delta}^{+}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.89}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\nabla}^{+}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Delta}^{\dagger}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\nabla}^{\dagger}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.90}$$

$$\tag{2.3.91}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Delta}^{+}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.92}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\nabla}^{+}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Delta}^{\dagger}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\nabla}^{\dagger}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.93}$$

$$\tag{2.3.94}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Delta_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\nabla_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Phi_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Delta^{+}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.95}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\nabla^{+}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Delta^{\dagger}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\nabla^{\dagger}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.3.96}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,\Gamma,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,\Gamma,-},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,\Gamma,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Phi}^{I}_{*,\Gamma,-},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{*,\Gamma,-}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{*,\Gamma,-}.$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^r_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^I_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^r_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \breve{\Phi}^I_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}.$$

Definition 2.3.16. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves}, \text{IndBanach}_{*} \tag{2.3.97}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.98}$$

$$\tag{2.3.99}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.100}$$

$$\tag{2.3.101}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}. \tag{2.3.102}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{*,\Gamma,-}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{*,\Gamma,-},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^r_{*,\Gamma,-}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^I_{*,\Gamma,-},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^r_{*,\Gamma,-}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^I_{*,\Gamma,-}.$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^r_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^I_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^r_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^I_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}.$$

Definition 2.3.17. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\mathrm{Quasicoherentsheaves}, \mathrm{Condensed}_{*} \tag{2.3.103}$$

where $*$ is one of the following spaces:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^{+}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.104}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^{+}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^{\dagger}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^{\dagger}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.105}$$

$$\tag{2.3.106}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^{+}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.107}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^{+}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^{\dagger}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^{\dagger}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.108}$$

$$\tag{2.3.109}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^{+}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.110}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^{+}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^{\dagger}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^{\dagger}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.3.111}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,\Gamma,-}, \underset{I}{\mathrm{homotopylimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,\Gamma,-},$$

$$\underset{r}{\mathrm{homotopycolimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,\Gamma,-}, \underset{I}{\mathrm{homotopylimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^{I}_{*,\Gamma,-},$$

$$\underset{r}{\mathrm{homotopycolimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{r}_{*,\Gamma,-}, \underset{I}{\mathrm{homotopylimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{I}_{*,\Gamma,-}.$$

$$\underset{r}{\mathrm{homotopycolimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}} \breve{\Phi}^{I}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{r}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}} \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{I}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 2.3.18. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Condensed}_{*} \tag{2.3.112}$$

where $*$ is one of the following spaces:

$$\underset{\mathrm{Spec}}{O}{}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.113}$$

$$\tag{2.3.114}$$

$$\underset{\mathrm{Spec}}{O}{}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.115}$$

$$\tag{2.3.116}$$

$$\underset{\mathrm{Spec}}{O}{}^{\mathrm{CS}}\Phi_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.117}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{O}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{O}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,-},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{O}{}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{O}{}^{\mathrm{CS}}\breve{\Phi}^I_{*,\Gamma,-},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{O}{}^{\mathrm{CS}}\Phi^r_{*,\Gamma,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{O}{}^{\mathrm{CS}}\Phi^I_{*,\Gamma,-}.$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{O}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{O}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{O}{}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \breve{\Phi}^I_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{O}{}^{\mathrm{CS}}\Phi^r_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{O}{}^{\mathrm{CS}}\Phi^I_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{2.3.118}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{2.3.119}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 2.3.19. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{IndBanach}_*. \tag{2.3.120}$$

Definition 2.3.20. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves}, \text{Perfectcomplex}, \text{IndBanach}_* \tag{2.3.121}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.122}$$

$$\tag{2.3.123}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.124}$$

$$\tag{2.3.125}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}. \tag{2.3.126}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{*,\Gamma,-}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{*,\Gamma,-},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^r_{*,\Gamma,-}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^I_{*,\Gamma,-},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^r_{*,\Gamma,-}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^I_{*,\Gamma,-}.$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^r_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^I_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^r_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^I_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}.$$

Definition 2.3.21. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves}, \text{Perfectcomplex}, \text{Condensed}_* \tag{2.3.127}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}^{+}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.128}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}^{+}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}^{\dagger}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}^{\dagger}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.129}$$

$$\tag{2.3.130}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}^{+}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.131}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\nabla}^{+}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}^{\dagger}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\nabla}^{\dagger}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.132}$$

$$\tag{2.3.133}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta^{+}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.134}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla^{+}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta^{\dagger}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla^{\dagger}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}. \tag{2.3.135}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{r}_{*,\Gamma,-}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{I}_{*,\Gamma,-},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^{r}_{*,\Gamma,-}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^{I}_{*,\Gamma,-},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{r}_{*,\Gamma,-}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{I}_{*,\Gamma,-}.$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{r}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{I}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^{r}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \breve{\Phi}^{I}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{r}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{I}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}.$$

Proposition 2.3.22. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Perfectcomplex, Condensed}_{*} \tag{2.3.136}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}_{*,\Gamma,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.137}$$

$$\tag{2.3.138}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.139}$$

$$\tag{2.3.140}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.141}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,-},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^I_{*,\Gamma,-},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{*,\Gamma,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{*,\Gamma,-}.$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \breve{\Phi}^I_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{*,\Gamma,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{2.3.142}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{2.3.143}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 2.3.23. Similar proposition holds for

$$\mathrm{Quasicoherentsheaves}, \mathrm{Perfectcomplex}, \mathrm{IndBanach}_*. \tag{2.3.144}$$

2.3.3 Frobenius Quasicoherent Prestacks III: Deformation in $(\infty,1)$-Ind-Banach Rings

Definition 2.3.24. We now consider the pro-étale site of $\mathrm{Spa}\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$, denote that by $*$. To be more accurate we replace one component for Γ with the pro-étale site of $\mathrm{Spa}\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$. And we treat then all the functor to be prestacks for this site. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power:

$$\widetilde{\Delta}_{*,\Gamma},\widetilde{\nabla}_{*,\Gamma},\widetilde{\Phi}_{*,\Gamma},\widetilde{\Delta}^+_{*,\Gamma},\widetilde{\nabla}^+_{*,\Gamma},\widetilde{\Delta}^\dagger_{*,\Gamma},\widetilde{\nabla}^\dagger_{*,\Gamma},\widetilde{\Phi}^r_{*,\Gamma},\widetilde{\Phi}^I_{*,\Gamma},$$

$$\breve{\Delta}_{*,\Gamma},\breve{\nabla}_{*,\Gamma},\breve{\Phi}_{*,\Gamma},\breve{\Delta}^+_{*,\Gamma},\breve{\nabla}^+_{*,\Gamma},\breve{\Delta}^\dagger_{*,\Gamma},\breve{\nabla}^\dagger_{*,\Gamma},\breve{\Phi}^r_{*,\Gamma},\breve{\Phi}^I_{*,\Gamma},$$

$$\Delta_{*,\Gamma},\nabla_{*,\Gamma},\Phi_{*,\Gamma},\Delta^+_{*,\Gamma},\nabla^+_{*,\Gamma},\Delta^\dagger_{*,\Gamma},\nabla^\dagger_{*,\Gamma},\Phi^r_{*,\Gamma},\Phi^I_{*,\Gamma}.$$

We now consider the following rings with $\square$ being a homotopy colimit

$$\underset{I}{\mathrm{homotopylimit}}\square_i \tag{2.3.145}$$

of $\mathbb{Q}_p\left\langle Y_1,...,Y_i\right\rangle, i=1,2,...$ in ∞-categories of simplicial ind-Banach rings in [BBBK]

$$\mathrm{SimplicialInd-BanachRings}_{\mathbb{Q}_p} \tag{2.3.146}$$

or animated analytic condensed commutative algebras in [CS2]

$$\mathrm{SimplicialAnalyticCondensed}_{\mathbb{Q}_p}. \tag{2.3.147}$$

Taking the product we have:

$$\widetilde{\Phi}_{*,\Gamma,\square},\widetilde{\Phi}^r_{*,\Gamma,\square},\widetilde{\Phi}^I_{*,\Gamma,\square},$$

$$\breve{\Phi}_{*,\Gamma,\square},\breve{\Phi}^r_{*,\Gamma,\square},\breve{\Phi}^I_{*,\Gamma,\square},$$

$$\Phi_{*,\Gamma,\square},\Phi^r_{*,\Gamma,\square},\Phi^I_{*,\Gamma,\square}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma:=\mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty,1)$-categories of $(\infty,1)$-modules.

Definition 2.3.25. First we consider the Bambozzi-Kremnizer spectrum $\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}_{*,\Gamma,\square},\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,\square},\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,\square}, \tag{2.3.148}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}_{*,\Gamma,\square},\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,\square},\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,\square}, \tag{2.3.149}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi_{*,\Gamma,\square}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^{r}_{*,\Gamma,\square}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^{I}_{*,\Gamma,\square}. \tag{2.3.150}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.151}$$

$$\tag{2.3.152}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.153}$$

$$\tag{2.3.154}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}. \tag{2.3.155}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^{r}_{*,\Gamma,\square}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^{I}_{*,\Gamma,\square},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^{r}_{*,\Gamma,\square}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^{I}_{*,\Gamma,\square},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^{r}_{*,\Gamma,\square}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^{I}_{*,\Gamma,\square}.$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^{r}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^{I}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^{r}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^{I}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^{r}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^{I}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 2.3.26. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Namely A will still as above as a Banach ring over $\mathbb{Q}_p$. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_{*,\Gamma}, \widetilde{\nabla}_{*,\Gamma}, \widetilde{\Phi}_{*,\Gamma}, \widetilde{\Delta}^{+}_{*,\Gamma}, \widetilde{\nabla}^{+}_{*,\Gamma}, \widetilde{\Delta}^{\dagger}_{*,\Gamma}, \widetilde{\nabla}^{\dagger}_{*,\Gamma}, \widetilde{\Phi}^{r}_{*,\Gamma}, \widetilde{\Phi}^{I}_{*,\Gamma},$$

$$\breve{\Delta}_{*,\Gamma}, \breve{\nabla}_{*,\Gamma}, \breve{\Phi}_{*,\Gamma}, \breve{\Delta}^{+}_{*,\Gamma}, \breve{\nabla}^{+}_{*,\Gamma}, \breve{\Delta}^{\dagger}_{*,\Gamma}, \breve{\nabla}^{\dagger}_{*,\Gamma}, \breve{\Phi}^{r}_{*,\Gamma}, \breve{\Phi}^{I}_{*,\Gamma},$$

$$\Delta_{*,\Gamma}, \nabla_{*,\Gamma}, \Phi_{*,\Gamma}, \Delta^{+}_{*,\Gamma}, \nabla^{+}_{*,\Gamma}, \Delta^{\dagger}_{*,\Gamma}, \nabla^{\dagger}_{*,\Gamma}, \Phi^{r}_{*,\Gamma}, \Phi^{I}_{*,\Gamma},$$

with A. Then we have the notations:

$$\widetilde{\Delta}_{*,\Gamma,\square}, \widetilde{\nabla}_{*,\Gamma,\square}, \widetilde{\Phi}_{*,\Gamma,\square}, \widetilde{\Delta}^+_{*,\Gamma,\square}, \widetilde{\nabla}^+_{*,\Gamma,\square}, \widetilde{\Delta}^\dagger_{*,\Gamma,\square}, \widetilde{\nabla}^\dagger_{*,\Gamma,\square}, \widetilde{\Phi}^r_{*,\Gamma,\square}, \widetilde{\Phi}^I_{*,\Gamma,\square},$$

$$\breve{\Delta}_{*,\Gamma,\square}, \breve{\nabla}_{*,\Gamma,\square}, \breve{\Phi}_{*,\Gamma,\square}, \breve{\Delta}^+_{*,\Gamma,\square}, \breve{\nabla}^+_{*,\Gamma,\square}, \breve{\Delta}^\dagger_{*,\Gamma,\square}, \breve{\nabla}^\dagger_{*,\Gamma,\square}, \breve{\Phi}^r_{*,\Gamma,\square}, \breve{\Phi}^I_{*,\Gamma,\square},$$

$$\Delta_{*,\Gamma,\square}, \nabla_{*,\Gamma,\square}, \Phi_{*,\Gamma,\square}, \Delta^+_{*,\Gamma,\square}, \nabla^+_{*,\Gamma,\square}, \Delta^\dagger_{*,\Gamma,\square}, \nabla^\dagger_{*,\Gamma,\square}, \Phi^r_{*,\Gamma,\square}, \Phi^I_{*,\Gamma,\square}.$$

Definition 2.3.27. First we consider the Clausen-Scholze spectrum $\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}_{*,\Gamma,\square}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}_{*,\Gamma,\square}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}_{*,\Gamma,\square}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}^+_{*,\Gamma,\square}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}^+_{*,\Gamma,\square}, \tag{2.3.156}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}^\dagger_{*,\Gamma,\square}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}^\dagger_{*,\Gamma,\square}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^r_{*,\Gamma,\square}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^I_{*,\Gamma,\square}, \tag{2.3.157}$$

$$\tag{2.3.158}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}_{*,\Gamma,\square}, \breve{\nabla}_{*,\Gamma,\square}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}_{*,\Gamma,\square}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}^+_{*,\Gamma,\square}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\nabla}^+_{*,\Gamma,\square}, \tag{2.3.159}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}^\dagger_{*,\Gamma,\square}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\nabla}^\dagger_{*,\Gamma,\square}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^r_{*,\Gamma,\square}, \breve{\Phi}^I_{*,\Gamma,\square}, \tag{2.3.160}$$

$$\tag{2.3.161}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta_{*,\Gamma,\square}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla_{*,\Gamma,\square}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi_{*,\Gamma,\square}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta^+_{*,\Gamma,\square}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla^+_{*,\Gamma,\square}, \tag{2.3.162}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta^\dagger_{*,\Gamma,\square}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla^\dagger_{*,\Gamma,\square}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{*,\Gamma,\square}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{*,\Gamma,\square}. \tag{2.3.163}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}^+_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.164}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}^+_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}^\dagger_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}^\dagger_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.165}$$

$$\tag{2.3.166}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}^+_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.167}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\nabla}^+_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}^\dagger_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\nabla}^\dagger_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.168}$$

$$\tag{2.3.169}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta^+_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.170}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla^+_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta^\dagger_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla^\dagger_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}. \tag{2.3.171}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^r_{*,\Gamma,\square}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^I_{*,\Gamma,\square},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^r_{*,\Gamma,\square}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^I_{*,\Gamma,\square},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{*,\Gamma,\square}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{*,\Gamma,\square}.$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^r_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^I_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^r_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \breve{\Phi}^I_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}.$$

Definition 2.3.28. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves}, \text{IndBanach}_* \tag{2.3.172}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.173}$$

$$\tag{2.3.174}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.175}$$

$$\tag{2.3.176}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}. \tag{2.3.177}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{*,\Gamma,\square}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{*,\Gamma,\square},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^r_{*,\Gamma,\square}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^I_{*,\Gamma,\square},$$

$$\underset{r}{\text{homotopycolimit}}\,\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^r_{*,\Gamma,\square}, \underset{I}{\text{homotopylimit}}\,\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^I_{*,\Gamma,\square}.$$

$$\underset{r}{\text{homotopycolimit}}\,\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\,\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\,\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^r_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\,\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^I_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\,\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^r_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\,\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^I_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}.$$

Definition 2.3.29. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Condensed}_* \tag{2.3.178}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}^+_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.179}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}^+_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}^\dagger_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}^\dagger_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.180}$$

$$\tag{2.3.181}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}^+_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.182}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\nabla}^+_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}^\dagger_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\nabla}^\dagger_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.183}$$

$$\tag{2.3.184}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta^+_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.185}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla^+_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta^\dagger_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla^\dagger_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}. \tag{2.3.186}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}}\,\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^r_{*,\Gamma,\square}, \underset{I}{\text{homotopylimit}}\,\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^I_{*,\Gamma,\square},$$

$$\underset{r}{\text{homotopycolimit}}\,\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^r_{*,\Gamma,\square}, \underset{I}{\text{homotopylimit}}\,\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^I_{*,\Gamma,\square},$$

$$\underset{r}{\text{homotopycolimit}}\,\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{*,\Gamma,\square}, \underset{I}{\text{homotopylimit}}\,\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{*,\Gamma,\square}.$$

$$\underset{r}{\text{homotopycolimit}}\,\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^r_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\,\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^I_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\,\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^r_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\,\breve{\Phi}^I_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\,\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\,\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}.$$

Proposition 2.3.30. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Condensed}_{*} \tag{2.3.187}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.188}$$

$$\tag{2.3.189}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.190}$$

$$\tag{2.3.191}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.192}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{r}_{*,\Gamma,\square}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{I}_{*,\Gamma,\square},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^{r}_{*,\Gamma,\square}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^{I}_{*,\Gamma,\square},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{r}_{*,\Gamma,\square}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{I}_{*,\Gamma,\square}.$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{r}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{I}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^{r}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \breve{\Phi}^{I}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{r}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{I}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{2.3.193}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{2.3.194}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 2.3.31. Similar proposition holds for

$$\mathrm{Quasicoherentsheaves}, \mathrm{IndBanach}_{*}. \tag{2.3.195}$$

Definition 2.3.32. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Perfectcomplex}, \mathrm{IndBanach}_{*} \tag{2.3.196}$$

where $*$ is one of the following spaces:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.197}$$

(2.3.198)

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.199}$$

(2.3.200)

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.3.201}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^{r}_{*,\Gamma,\square}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^{I}_{*,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^{r}_{*,\Gamma,\square}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^{I}_{*,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^{r}_{*,\Gamma,\square}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^{I}_{*,\Gamma,\square}.$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^{r}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^{I}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^{r}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^{I}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^{r}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^{I}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 2.3.33. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Perfectcomplex, Condensed}_{*} \tag{2.3.202}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}^{+}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.203}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}^{+}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}^{\dagger}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}^{\dagger}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.204}$$

$$\tag{2.3.205}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}^{+}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.206}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\nabla}^{+}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}^{\dagger}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\nabla}^{\dagger}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.207}$$

$$\tag{2.3.208}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta^{+}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \tag{2.3.209}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla^{+}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta^{\dagger}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla^{\dagger}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}. \tag{2.3.210}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{r}_{*,\Gamma,\square}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{I}_{*,\Gamma,\square},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^{r}_{*,\Gamma,\square}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^{I}_{*,\Gamma,\square},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{r}_{*,\Gamma,\square}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{I}_{*,\Gamma,\square}.$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{r}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{I}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^{r}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \breve{\Phi}^{I}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{r}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\ \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{I}_{*,\Gamma,\square}/\text{Fro}^{\mathbb{Z}}.$$

Proposition 2.3.34. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Perfectcomplex, Condensed}_{*} \tag{2.3.211}$$

where $*$ is one of the following spaces:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.212}$$

(2.3.213)

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.214}$$

(2.3.215)

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.3.216}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,\square}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,\square}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^I_{*,\Gamma,\square},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{*,\Gamma,\square}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{*,\Gamma,\square}.$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \breve{\Phi}^I_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{*,\Gamma,\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{2.3.217}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{2.3.218}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 2.3.35. Similar proposition holds for

$$\mathrm{Quasicoherentsheaves}, \mathrm{Perfectcomplex}, \mathrm{IndBanach}_*. \tag{2.3.219}$$

2.4 Univariate Hodge Iwasawa Prestacks by Deformation

This chapter follows closely [T1], [T2], [T3], [T4], [T5], [T6], [KPX], [KP], [KL1], [KL2], [BK], [BBBK], [BBM], [KKM], [CS1], [CS2], [LBV].

2.4.1 Frobenius Quasicoherent Prestacks I

Definition 2.4.1. First we consider the Bambozzi-Kremnizer spectrum $\underset{\mathrm{Spec}}{O}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\underset{\mathrm{Spec}}{O}^{\mathrm{BK}}\widetilde{\Phi}_{*,A}, \underset{\mathrm{Spec}}{O}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,A}, \underset{\mathrm{Spec}}{O}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,A}, \tag{2.4.1}$$

$$\underset{\mathrm{Spec}}{O}^{\mathrm{BK}}\breve{\Phi}_{*,A}, \underset{\mathrm{Spec}}{O}^{\mathrm{BK}}\breve{\Phi}^r_{*,A}, \underset{\mathrm{Spec}}{O}^{\mathrm{BK}}\breve{\Phi}^I_{*,A}, \tag{2.4.2}$$

$$\underset{\mathrm{Spec}}{O}^{\mathrm{BK}}\Phi_{*,A}, \underset{\mathrm{Spec}}{O}^{\mathrm{BK}}\Phi^r_{*,A}, \underset{\mathrm{Spec}}{O}^{\mathrm{BK}}\Phi^I_{*,A}. \tag{2.4.3}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\underset{\mathrm{Spec}}{O}^{\mathrm{BK}}\widetilde{\Phi}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.4}$$

$$\tag{2.4.5}$$

$$\underset{\mathrm{Spec}}{O}^{\mathrm{BK}}\breve{\Phi}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.6}$$

$$\tag{2.4.7}$$

$$\underset{\mathrm{Spec}}{O}^{\mathrm{BK}}\Phi_{*,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.4.8}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{O}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{O}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,A},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{O}^{\mathrm{BK}}\breve{\Phi}^r_{*,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{O}^{\mathrm{BK}}\breve{\Phi}^I_{*,A},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{O}^{\mathrm{BK}}\Phi^r_{*,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{O}^{\mathrm{BK}}\Phi^I_{*,A}.$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{O}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{O}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{O}^{\mathrm{BK}}\breve{\Phi}^r_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{O}^{\mathrm{BK}}\breve{\Phi}^I_{*,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{O}^{\mathrm{BK}}\Phi^r_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{O}^{\mathrm{BK}}\Phi^I_{*,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 2.4.2. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Namely A will still as above as a Banach ring over $\mathbb{Q}_p$. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_*, \widetilde{\nabla}_*, \widetilde{\Phi}_*, \widetilde{\Delta}^+_*, \widetilde{\nabla}^+_*, \widetilde{\Delta}^\dagger_*, \widetilde{\nabla}^\dagger_*, \widetilde{\Phi}^r_*, \widetilde{\Phi}^I_*,$$

$$\breve{\Delta}_*, \breve{\nabla}_*, \breve{\Phi}_*, \breve{\Delta}^+_*, \breve{\nabla}^+_*, \breve{\Delta}^\dagger_*, \breve{\nabla}^\dagger_*, \breve{\Phi}^r_*, \breve{\Phi}^I_*,$$

$$\Delta_*, \nabla_*, \Phi_*, \Delta^+_*, \nabla^+_*, \Delta^\dagger_*, \nabla^\dagger_*, \Phi^r_*, \Phi^I_*,$$

with A. Then we have the notations:

$$\widetilde{\Delta}_{*,A}, \widetilde{\nabla}_{*,A}, \widetilde{\Phi}_{*,A}, \widetilde{\Delta}^+_{*,A}, \widetilde{\nabla}^+_{*,A}, \widetilde{\Delta}^\dagger_{*,A}, \widetilde{\nabla}^\dagger_{*,A}, \widetilde{\Phi}^r_{*,A}, \widetilde{\Phi}^I_{*,A},$$

$$\breve{\Delta}_{*,A}, \breve{\nabla}_{*,A}, \breve{\Phi}_{*,A}, \breve{\Delta}^+_{*,A}, \breve{\nabla}^+_{*,A}, \breve{\Delta}^\dagger_{*,A}, \breve{\nabla}^\dagger_{*,A}, \breve{\Phi}^r_{*,A}, \breve{\Phi}^I_{*,A},$$

$$\Delta_{*,A}, \nabla_{*,A}, \Phi_{*,A}, \Delta^+_{*,A}, \nabla^+_{*,A}, \Delta^\dagger_{*,A}, \nabla^\dagger_{*,A}, \Phi^r_{*,A}, \Phi^I_{*,A}.$$

Definition 2.4.3. First we consider the Clausen-Scholze spectrum $\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}_{*,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}_{*,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}_{*,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,A}, \tag{2.4.9}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,A}, \tag{2.4.10}$$

$$\tag{2.4.11}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}_{*,A}, \breve{\nabla}_{*,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{*,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^+_{*,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^+_{*,A}, \tag{2.4.12}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{*,A}, \breve{\Phi}^I_{*,A}, \tag{2.4.13}$$

$$\tag{2.4.14}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta_{*,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla_{*,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{*,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^+_{*,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^+_{*,A}, \tag{2.4.15}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^\dagger_{*,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^\dagger_{*,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{*,A}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{*,A}. \tag{2.4.16}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.17}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.18}$$

$$\tag{2.4.19}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^{+}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.20}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^{+}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^{\dagger}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^{\dagger}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.21}$$

$$\tag{2.4.22}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^{+}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.23}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^{+}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^{\dagger}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^{\dagger}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.4.24}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,A},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^{I}_{*,A},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{r}_{*,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{I}_{*,A}.$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \breve{\Phi}^{I}_{*,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{r}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{I}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 2.4.4. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\mathrm{Quasicoherentpresheaves}, \mathrm{IndBanach}_{*} \tag{2.4.25}$$

where $*$ is one of the following spaces:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.26}$$

$$\tag{2.4.27}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.28}$$

$$\tag{2.4.29}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi_{*,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.4.30}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{*,A}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{*,A},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^r_{*,A}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^I_{*,A},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^r_{*,A}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^I_{*,A}.$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{*,A}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{*,A}/\text{Fro}^{\mathbb{Z}},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^r_{*,A}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^I_{*,A}/\text{Fro}^{\mathbb{Z}},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^r_{*,A}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^I_{*,A}/\text{Fro}^{\mathbb{Z}}.$$

Definition 2.4.5. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Condensed}_* \tag{2.4.31}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}_{*,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}_{*,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}_{*,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}^+_{*,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.4.32}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}^+_{*,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}^\dagger_{*,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}^\dagger_{*,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.4.33}$$

$$\tag{2.4.34}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}_{*,A}/\text{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}_{*,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}^+_{*,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.4.35}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\nabla}^+_{*,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}^\dagger_{*,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\nabla}^\dagger_{*,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.4.36}$$

$$\tag{2.4.37}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta_{*,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla_{*,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi_{*,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta^+_{*,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.4.38}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla^+_{*,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta^\dagger_{*,A}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla^\dagger_{*,A}/\text{Fro}^{\mathbb{Z}}. \tag{2.4.39}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^r_{*,A}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^I_{*,A},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^r_{*,A}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^I_{*,A},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{*,A}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{*,A}.$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^r_{*,A}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^I_{*,A}/\text{Fro}^{\mathbb{Z}},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^r_{*,A}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \breve{\Phi}^I_{*,A}/\text{Fro}^{\mathbb{Z}},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{*,A}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{*,A}/\text{Fro}^{\mathbb{Z}}.$$

Proposition 2.4.6. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Condensed}_* \tag{2.4.40}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}_{*,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.4.41}$$

$$\tag{2.4.42}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}_{*,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.4.43}$$

$$\tag{2.4.44}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi_{*,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.4.45}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^r_{*,A}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^I_{*,A},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^r_{*,A}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^I_{*,A},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{*,A}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{*,A}.$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^r_{*,A}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^I_{*,A}/\text{Fro}^{\mathbb{Z}},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^r_{*,A}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \breve{\Phi}^I_{*,A}/\text{Fro}^{\mathbb{Z}},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{*,A}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{*,A}/\text{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\text{homotopycolimit}_r M_r, \tag{2.4.46}$$

$$\text{homotopylimit}_I M_I, \tag{2.4.47}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 2.4.7. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{IndBanach}_*. \tag{2.4.48}$$

Definition 2.4.8. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves}, \text{Perfectcomplex}, \text{IndBanach}_* \tag{2.4.49}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}_{*,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.4.50}$$

(2.4.51)

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}_{*,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.4.52}$$

(2.4.53)

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi_{*,A}/\text{Fro}^{\mathbb{Z}}. \tag{2.4.54}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{*,A}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{*,A},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^r_{*,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^I_{*,A},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^r_{*,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^I_{*,A}.$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^r_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^I_{*,A}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^r_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^I_{*,A}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 2.4.9. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\mathrm{Quasicoherentsheaves, Perfectcomplex, Condensed}_* \tag{2.4.55}$$

where $*$ is one of the following spaces:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.56}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.57}$$

$$\tag{2.4.58}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^+_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.59}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^+_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.60}$$

$$\tag{2.4.61}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^+_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.62}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^+_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^\dagger_{*,A}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^\dagger_{*,A}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.4.63}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,A},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{*,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^I_{*,A},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{*,A}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{*,A}.$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^r_{*,A}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^I_{*,A}/\text{Fro}^{\mathbb{Z}},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^r_{*,A}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \breve{\Phi}^I_{*,A}/\text{Fro}^{\mathbb{Z}},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{*,A}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{*,A}/\text{Fro}^{\mathbb{Z}}.$$

Proposition 2.4.10. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Perfectcomplex, Condensed}_* \tag{2.4.64}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}_{*,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.4.65}$$

$$\tag{2.4.66}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}_{*,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.4.67}$$

$$\tag{2.4.68}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi_{*,A}/\text{Fro}^{\mathbb{Z}}, \tag{2.4.69}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^r_{*,A}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^I_{*,A},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^r_{*,A}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^I_{*,A},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{*,A}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{*,A}.$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^r_{*,A}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^I_{*,A}/\text{Fro}^{\mathbb{Z}},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^r_{*,A}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \breve{\Phi}^I_{*,A}/\text{Fro}^{\mathbb{Z}},$$

$$\text{homotopycolimit}_r \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{*,A}/\text{Fro}^{\mathbb{Z}}, \text{homotopylimit}_I \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{*,A}/\text{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\mathrm{homotopycolimit}_{r} M_r, \tag{2.4.70}$$

$$\mathrm{homotopylimit}_{I} M_I, \tag{2.4.71}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 2.4.11. Similar proposition holds for

$$\mathrm{Quasicoherentsheaves}, \mathrm{Perfectcomplex}, \mathrm{IndBanach}_*. \tag{2.4.72}$$

2.4.2 Frobenius Quasicoherent Prestacks II: Deformation in Banach Rings

Definition 2.4.12. We now consider the pro-étale site of $\mathrm{Spa}\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$, denote that by $*$. To be more accurate we replace one component for Γ with the pro-étale site of $\mathrm{Spa}\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$. And we treat then all the functor to be prestacks for this site. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power[3]:

$$\widetilde{\Delta}_*, \widetilde{\nabla}_*, \widetilde{\Phi}_*, \widetilde{\Delta}^+_*, \widetilde{\nabla}^+_*, \widetilde{\Delta}^\dagger_*, \widetilde{\nabla}^\dagger_*, \widetilde{\Phi}^r_*, \widetilde{\Phi}^I_*,$$

$$\breve{\Delta}_*, \breve{\nabla}_*, \breve{\Phi}_*, \breve{\Delta}^+_*, \breve{\nabla}^+_*, \breve{\Delta}^\dagger_*, \breve{\nabla}^\dagger_*, \breve{\Phi}^r_*, \breve{\Phi}^I_*,$$

$$\Delta_*, \nabla_*, \Phi_*, \Delta^+_*, \nabla^+_*, \Delta^\dagger_*, \nabla^\dagger_*, \Phi^r_*, \Phi^I_*.$$

We now consider the following rings with $-$ being any deforming Banach ring over $\mathbb{Q}_p$. Taking the product we have:

$$\widetilde{\Phi}_{*,-}, \widetilde{\Phi}^r_{*,-}, \widetilde{\Phi}^I_{*,-},$$

$$\breve{\Phi}_{*,-}, \breve{\Phi}^r_{*,-}, \breve{\Phi}^I_{*,-},$$

$$\Phi_{*,-}, \Phi^r_{*,-}, \Phi^I_{*,-}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma := \mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty,1)$-categories of $(\infty,1)$-modules.

Definition 2.4.13. First we consider the Bambozzi-Kremnizer spectrum $\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}_{*,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,-}, \tag{2.4.73}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}_{*,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^r_{*,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^I_{*,-}, \tag{2.4.74}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi_{*,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^r_{*,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^I_{*,-}. \tag{2.4.75}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.76}$$

$$\tag{2.4.77}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.78}$$

$$\tag{2.4.79}$$

[3]Here $|\Gamma| = 1$.

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi_{*,-}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.4.80}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^{r}_{*,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^{I}_{*,-},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^{r}_{*,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^{I}_{*,-},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^{r}_{*,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^{I}_{*,-}.$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^{r}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^{I}_{*,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^{r}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^{I}_{*,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^{r}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^{I}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 2.4.14. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Namely A will still as above as a Banach ring over $\mathbb{Q}_p$. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_*, \widetilde{\nabla}_*, \widetilde{\Phi}_*, \widetilde{\Delta}^+_*, \widetilde{\nabla}^+_*, \widetilde{\Delta}^\dagger_*, \widetilde{\nabla}^\dagger_*, \widetilde{\Phi}^r_*, \widetilde{\Phi}^I_*,$$

$$\breve{\Delta}_*, \breve{\nabla}_*, \breve{\Phi}_*, \breve{\Delta}^+_*, \breve{\nabla}^+_*, \breve{\Delta}^\dagger_*, \breve{\nabla}^\dagger_*, \breve{\Phi}^r_*, \breve{\Phi}^I_*,$$

$$\Delta_*, \nabla_*, \Phi_*, \Delta^+_*, \nabla^+_*, \Delta^\dagger_*, \nabla^\dagger_*, \Phi^r_*, \Phi^I_*,$$

with A. Then we have the notations:

$$\widetilde{\Delta}_{*,-}, \widetilde{\nabla}_{*,-}, \widetilde{\Phi}_{*,-}, \widetilde{\Delta}^+_{*,-}, \widetilde{\nabla}^+_{*,-}, \widetilde{\Delta}^\dagger_{*,-}, \widetilde{\nabla}^\dagger_{*,-}, \widetilde{\Phi}^r_{*,-}, \widetilde{\Phi}^I_{*,-},$$

$$\breve{\Delta}_{*,-}, \breve{\nabla}_{*,-}, \breve{\Phi}_{*,-}, \breve{\Delta}^+_{*,-}, \breve{\nabla}^+_{*,-}, \breve{\Delta}^\dagger_{*,-}, \breve{\nabla}^\dagger_{*,-}, \breve{\Phi}^r_{*,-}, \breve{\Phi}^I_{*,-},$$

$$\Delta_{*,-}, \nabla_{*,-}, \Phi_{*,-}, \Delta^+_{*,-}, \nabla^+_{*,-}, \Delta^\dagger_{*,-}, \nabla^\dagger_{*,-}, \Phi^r_{*,-}, \Phi^I_{*,-}.$$

Definition 2.4.15. First we consider the Clausen-Scholze spectrum $\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}_{*,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}_{*,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}_{*,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^{+}_{*,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^{+}_{*,-}, \tag{2.4.81}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^{\dagger}_{*,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^{\dagger}_{*,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,-}, \tag{2.4.82}$$

$$\tag{2.4.83}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}_{*,-}, \breve{\nabla}_{*,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{*,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^{+}_{*,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^{+}_{*,-}, \tag{2.4.84}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^{\dagger}_{*,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^{\dagger}_{*,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,-}, \breve{\Phi}^{I}_{*,-}, \tag{2.4.85}$$

$$\tag{2.4.86}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta_{*,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla_{*,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{*,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^{+}_{*,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^{+}_{*,-}, \tag{2.4.87}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^{\dagger}_{*,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^{\dagger}_{*,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{r}_{*,-}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{I}_{*,-}. \tag{2.4.88}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^{+}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.89}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^{+}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Delta}^{\dagger}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\nabla}^{\dagger}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.90}$$

$$\tag{2.4.91}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^{+}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.92}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^{+}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Delta}^{\dagger}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\nabla}^{\dagger}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.93}$$

$$\tag{2.4.94}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^{+}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.95}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^{+}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Delta^{\dagger}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\nabla^{\dagger}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.4.96}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,-}, \underset{I}{\mathrm{homotopylimit}}\, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,-},$$

$$\underset{r}{\mathrm{homotopycolimit}}\, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,-}, \underset{I}{\mathrm{homotopylimit}}\, \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^{I}_{*,-},$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{*,-}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{*,-}.$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^r_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{homotopylimit}_I \breve{\Phi}^I_{*,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^r_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^I_{*,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 2.4.16. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\mathrm{Quasicoherentpresheaves}, \mathrm{IndBanach}_* \tag{2.4.97}$$

where $*$ is one of the following spaces:

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.98}$$

$$\tag{2.4.99}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.100}$$

$$\tag{2.4.101}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi_{*,-}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.4.102}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,-}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,-},$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^r_{*,-}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^I_{*,-},$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^r_{*,-}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^I_{*,-}.$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^r_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\breve{\Phi}^I_{*,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\mathrm{homotopycolimit}_r \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^r_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{homotopylimit}_I \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{BK}}\Phi^I_{*,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 2.4.17. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Condensed}_{*} \tag{2.4.103}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}_{*,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}_{*,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}_{*,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}^{+}_{*,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.4.104}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}^{+}_{*,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Delta}^{\dagger}_{*,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\nabla}^{\dagger}_{*,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.4.105}$$

$$\tag{2.4.106}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}_{*,-}/\text{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}_{*,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}^{+}_{*,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.4.107}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\nabla}^{+}_{*,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Delta}^{\dagger}_{*,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\nabla}^{\dagger}_{*,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.4.108}$$

$$\tag{2.4.109}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta_{*,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla_{*,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi_{*,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta^{+}_{*,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.4.110}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla^{+}_{*,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Delta^{\dagger}_{*,-}/\text{Fro}^{\mathbb{Z}}, \underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\nabla^{\dagger}_{*,-}/\text{Fro}^{\mathbb{Z}}. \tag{2.4.111}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}}\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{r}_{*,-}, \underset{I}{\text{homotopylimit}}\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{I}_{*,-},$$

$$\underset{r}{\text{homotopycolimit}}\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^{r}_{*,-}, \underset{I}{\text{homotopylimit}}\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^{I}_{*,-},$$

$$\underset{r}{\text{homotopycolimit}}\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{r}_{*,-}, \underset{I}{\text{homotopylimit}}\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{I}_{*,-}.$$

$$\underset{r}{\text{homotopycolimit}}\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{r}_{*,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^{I}_{*,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^{r}_{*,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\breve{\Phi}^{I}_{*,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{r}_{*,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^{I}_{*,-}/\text{Fro}^{\mathbb{Z}}.$$

Proposition 2.4.18. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Condensed}_{*} \tag{2.4.112}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}_{*,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.4.113}$$

$$\tag{2.4.114}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}_{*,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.4.115}$$

$$\tag{2.4.116}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi_{*,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.4.117}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}}\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^r_{*,-}, \underset{I}{\text{homotopylimit}}\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^I_{*,-},$$

$$\underset{r}{\text{homotopycolimit}}\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^r_{*,-}, \underset{I}{\text{homotopylimit}}\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^I_{*,-},$$

$$\underset{r}{\text{homotopycolimit}}\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{*,-}, \underset{I}{\text{homotopylimit}}\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{*,-}.$$

$$\underset{r}{\text{homotopycolimit}}\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^r_{*,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\widetilde{\Phi}^I_{*,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\breve{\Phi}^r_{*,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\,\breve{\Phi}^I_{*,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}}\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^r_{*,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}}\underset{\text{Spec}}{\mathcal{O}}{}^{\text{CS}}\Phi^I_{*,-}/\text{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\, M_r, \tag{2.4.118}$$

$$\underset{I}{\text{homotopylimit}}\, M_I, \tag{2.4.119}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 2.4.19. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{IndBanach}_*. \tag{2.4.120}$$

Definition 2.4.20. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves}, \text{Perfectcomplex}, \text{IndBanach}_* \tag{2.4.121}$$

where $*$ is one of the following spaces:

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}_{*,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.4.122}$$

$$\tag{2.4.123}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}_{*,-}/\text{Fro}^{\mathbb{Z}}, \tag{2.4.124}$$

$$\tag{2.4.125}$$

$$\underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi_{*,-}/\text{Fro}^{\mathbb{Z}}. \tag{2.4.126}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{*,-}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{*,-},$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^r_{*,-}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^I_{*,-},$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^r_{*,-}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^I_{*,-}.$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^r_{*,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\widetilde{\Phi}^I_{*,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^r_{*,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\breve{\Phi}^I_{*,-}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopycolimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^r_{*,-}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopylimit}} \underset{\text{Spec}}{\mathcal{O}}{}^{\text{BK}}\Phi^I_{*,-}/\text{Fro}^{\mathbb{Z}}.$$

Definition 2.4.21. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves}, \text{Perfectcomplex}, \text{Condensed}_* \tag{2.4.127}$$

where $*$ is one of the following spaces:

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Delta}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\nabla}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Phi}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Delta}^{+}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.128}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\nabla}^{+}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Delta}^{\dagger}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\nabla}^{\dagger}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.129}$$

$$\tag{2.4.130}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Delta}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Phi}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Delta}^{+}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.131}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\nabla}^{+}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Delta}^{\dagger}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\nabla}^{\dagger}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.132}$$

$$\tag{2.4.133}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Delta_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\nabla_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Phi_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Delta^{+}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.134}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\nabla^{+}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Delta^{\dagger}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\nabla^{\dagger}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}. \tag{2.4.135}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,-},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Phi}^{I}_{*,-},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Phi^{r}_{*,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Phi^{I}_{*,-}.$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \breve{\Phi}^{I}_{*,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Phi^{r}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\Phi^{I}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 2.4.22. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Perfectcomplex}, \mathrm{Condensed}_{*} \tag{2.4.136}$$

where $*$ is one of the following spaces:

$$\underset{\mathrm{Spec}}{\mathcal{O}}^{\mathrm{CS}}\widetilde{\Phi}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.137}$$

$$\tag{2.4.138}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.139}$$

$$\tag{2.4.140}$$

$$\underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \tag{2.4.141}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,-},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^{I}_{*,-},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{r}_{*,-}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{I}_{*,-}.$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \breve{\Phi}^{I}_{*,-}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopycolimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{r}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopylimit}}\ \underset{\mathrm{Spec}}{\mathcal{O}}{}^{\mathrm{CS}}\Phi^{I}_{*,-}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{2.4.142}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{2.4.143}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 2.4.23. Similar proposition holds for

$$\mathrm{Quasicoherentsheaves}, \mathrm{Perfectcomplex}, \mathrm{IndBanach}_{*}. \tag{2.4.144}$$

Chapter 3

Over General Stacks

3.1 Over Preadic Spaces

This chapter follows closely [T1], [T2], [T3], [KPX], [KP], [KL1], [KL2], [BK], [BBBK], [BBM], [KKM], [CS1], [CS2], [CKZ], [PZ], [BCM], [LBV]. All the preadic spaces will be those defined as in [Hu1], [Hu2], [KL1], [KL2], [SW], while we regard them as the corresponding ∞-ringed ∞-toposes in the sense of Bambozzi-Kremnizer and Clausen-Scholze as in [BK], [BBBK], [BBM], [KKM], [CS1], [CS2] by directly animating the corresponding presheaves in the previous categories to reach the enhancement.

3.1.1 Multivariate Hodge Iwasawa Modules

Frobenius Quasicoherent Modules I

Definition 3.1.1. Let ψ be a toric tower over $\mathbb{Q}_p$ as in [KL2, Chapter 7] with base $\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power:

$$\widetilde{\Delta}_{\psi,\Gamma}, \widetilde{\nabla}_{\psi,\Gamma}, \widetilde{\Phi}_{\psi,\Gamma}, \widetilde{\Delta}^+_{\psi,\Gamma}, \widetilde{\nabla}^+_{\psi,\Gamma}, \widetilde{\Delta}^\dagger_{\psi,\Gamma}, \widetilde{\nabla}^\dagger_{\psi,\Gamma}, \widetilde{\Phi}^r_{\psi,\Gamma}, \widetilde{\Phi}^I_{\psi,\Gamma},$$

$$\breve{\Delta}_{\psi,\Gamma}, \breve{\nabla}_{\psi,\Gamma}, \breve{\Phi}_{\psi,\Gamma}, \breve{\Delta}^+_{\psi,\Gamma}, \breve{\nabla}^+_{\psi,\Gamma}, \breve{\Delta}^\dagger_{\psi,\Gamma}, \breve{\nabla}^\dagger_{\psi,\Gamma}, \breve{\Phi}^r_{\psi,\Gamma}, \breve{\Phi}^I_{\psi,\Gamma},$$

$$\Delta_{\psi,\Gamma}, \nabla_{\psi,\Gamma}, \Phi_{\psi,\Gamma}, \Delta^+_{\psi,\Gamma}, \nabla^+_{\psi,\Gamma}, \Delta^\dagger_{\psi,\Gamma}, \nabla^\dagger_{\psi,\Gamma}, \Phi^r_{\psi,\Gamma}, \Phi^I_{\psi,\Gamma}.$$

Here in the following we have X a preadic space over $\mathbb{Q}_p$. Taking the product we have:

$$\widetilde{\Phi}_{\psi,\Gamma,X}, \widetilde{\Phi}^r_{\psi,\Gamma,X}, \widetilde{\Phi}^I_{\psi,\Gamma,X},$$

$$\breve{\Phi}_{\psi,\Gamma,X}, \breve{\Phi}^r_{\psi,\Gamma,X}, \breve{\Phi}^I_{\psi,\Gamma,X},$$

$$\Phi_{\psi,\Gamma,X}, \Phi^r_{\psi,\Gamma,X}, \Phi^I_{\psi,\Gamma,X}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma := \mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty,1)$-categories of $(\infty,1)$-modules.

Definition 3.1.2. First we consider the Bambozzi-Kremnizer spectrum $\mathrm{Spec}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,X}, \tag{3.1.1}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,X}, \tag{3.1.2}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,X}. \tag{3.1.3}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.4}$$
$$\tag{3.1.5}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.6}$$
$$\tag{3.1.7}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.8}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,X}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 3.1.3. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_{\psi,\Gamma}, \widetilde{\nabla}_{\psi,\Gamma}, \widetilde{\Phi}_{\psi,\Gamma}, \widetilde{\Delta}^+_{\psi,\Gamma}, \widetilde{\nabla}^+_{\psi,\Gamma}, \widetilde{\Delta}^\dagger_{\psi,\Gamma}, \widetilde{\nabla}^\dagger_{\psi,\Gamma}, \widetilde{\Phi}^r_{\psi,\Gamma}, \widetilde{\Phi}^I_{\psi,\Gamma},$$

$$\breve{\Delta}_{\psi,\Gamma}, \breve{\nabla}_{\psi,\Gamma}, \breve{\Phi}_{\psi,\Gamma}, \breve{\Delta}^+_{\psi,\Gamma}, \breve{\nabla}^+_{\psi,\Gamma}, \breve{\Delta}^\dagger_{\psi,\Gamma}, \breve{\nabla}^\dagger_{\psi,\Gamma}, \breve{\Phi}^r_{\psi,\Gamma}, \breve{\Phi}^I_{\psi,\Gamma},$$

$$\Delta_{\psi,\Gamma}, \nabla_{\psi,\Gamma}, \Phi_{\psi,\Gamma}, \Delta^+_{\psi,\Gamma}, \nabla^+_{\psi,\Gamma}, \Delta^\dagger_{\psi,\Gamma}, \nabla^\dagger_{\psi,\Gamma}, \Phi^r_{\psi,\Gamma}, \Phi^I_{\psi,\Gamma},$$

with X. Then we have the notations:

$$\widetilde{\Delta}_{\psi,\Gamma,X}, \widetilde{\nabla}_{\psi,\Gamma,X}, \widetilde{\Phi}_{\psi,\Gamma,X}, \widetilde{\Delta}^+_{\psi,\Gamma,X}, \widetilde{\nabla}^+_{\psi,\Gamma,X}, \widetilde{\Delta}^\dagger_{\psi,\Gamma,X}, \widetilde{\nabla}^\dagger_{\psi,\Gamma,X}, \widetilde{\Phi}^r_{\psi,\Gamma,X}, \widetilde{\Phi}^I_{\psi,\Gamma,X},$$

$$\breve{\Delta}_{\psi,\Gamma,X}, \breve{\nabla}_{\psi,\Gamma,X}, \breve{\Phi}_{\psi,\Gamma,X}, \breve{\Delta}^+_{\psi,\Gamma,X}, \breve{\nabla}^+_{\psi,\Gamma,X}, \breve{\Delta}^\dagger_{\psi,\Gamma,X}, \breve{\nabla}^\dagger_{\psi,\Gamma,X}, \breve{\Phi}^r_{\psi,\Gamma,X}, \breve{\Phi}^I_{\psi,\Gamma,X},$$

$$\Delta_{\psi,\Gamma,X}, \nabla_{\psi,\Gamma,X}, \Phi_{\psi,\Gamma,X}, \Delta^+_{\psi,\Gamma,X}, \nabla^+_{\psi,\Gamma,X}, \Delta^\dagger_{\psi,\Gamma,X}, \nabla^\dagger_{\psi,\Gamma,X}, \Phi^r_{\psi,\Gamma,X}, \Phi^I_{\psi,\Gamma,X}.$$

Definition 3.1.4. First we consider the Clausen-Scholze spectrum $\mathrm{Spec}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,X}, \tag{3.1.9}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X}, \tag{3.1.10}$$

$$\tag{3.1.11}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,X}, \breve{\nabla}_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,X}, \tag{3.1.12}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X}, \breve{\Phi}^I_{\psi,\Gamma,X}, \tag{3.1.13}$$

$$\tag{3.1.14}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,X}, \tag{3.1.15}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X}. \tag{3.1.16}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.17}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.18}$$

$$\tag{3.1.19}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.20}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.21}$$

$$\tag{3.1.22}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.23}$$
$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.24}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r,\bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},\underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 3.1.5. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\mathrm{Quasicoherentpresheaves},\mathrm{IndBanach}_{*} \tag{3.1.25}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.26}$$
$$\tag{3.1.27}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.28}$$
$$\tag{3.1.29}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.30}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r,\bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,X},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,X},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,X},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,X}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 3.1.6. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Condensed}_* \tag{3.1.31}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.32}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.33}$$

$$\tag{3.1.34}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.35}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.36}$$

$$\tag{3.1.37}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.38}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.39}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,X},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 3.1.7. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Condensed}_* \tag{3.1.40}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.41}$$

(3.1.42)

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.43}$$

(3.1.44)

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.45}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients[1]. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\text{homotopylimit}_r\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X}, \text{homotopycolimit}_I\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X},$$

$$\text{homotopylimit}_r\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X}, \text{homotopycolimit}_I\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,X},$$

$$\text{homotopylimit}_r\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X}, \text{homotopycolimit}_I\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X}.$$

$$\text{homotopylimit}_r\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \text{homotopycolimit}_I\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\text{homotopylimit}_r\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \text{homotopycolimit}_I\ \breve{\Phi}^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\text{homotopylimit}_r\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \text{homotopycolimit}_I\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding

[1]When we are considering at current generality the corresponding deformation over general preadic stacks, we will further consider the corresponding Frobenius modules over period sheaves (not rings though we talk about rings if it is not too confusing) carrying coefficients in preadic stacks.

quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\text{homotopycolimit}_r M_r, \tag{3.1.46}$$

$$\text{homotopylimit}_I M_I, \tag{3.1.47}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 3.1.8. Similar proposition holds for

$$\text{Quasicoherentsheaves, IndBanach}_*. \tag{3.1.48}$$

Definition 3.1.9. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves, Perfectcomplex, IndBanach}_* \tag{3.1.49}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{BK}}\widetilde{\Phi}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.50}$$

$$\tag{3.1.51}$$

$$\text{Spec}^{\text{BK}}\breve{\Phi}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.52}$$

$$\tag{3.1.53}$$

$$\text{Spec}^{\text{BK}}\Phi_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}. \tag{3.1.54}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\text{homotopylimit}_r \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,X}, \text{homotopycolimit}_I \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,X},$$

$$\text{homotopylimit}_r \text{Spec}^{\text{BK}}\breve{\Phi}^r_{\psi,\Gamma,X}, \text{homotopycolimit}_I \text{Spec}^{\text{BK}}\breve{\Phi}^I_{\psi,\Gamma,X},$$

$$\text{homotopylimit}_r \text{Spec}^{\text{BK}}\Phi^r_{\psi,\Gamma,X}, \text{homotopycolimit}_I \text{Spec}^{\text{BK}}\Phi^I_{\psi,\Gamma,X}.$$

$$\text{homotopylimit}_r \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{homotopycolimit}_I \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}},$$

$$\text{homotopylimit}_r \text{Spec}^{\text{BK}}\breve{\Phi}^r_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{homotopycolimit}_I \text{Spec}^{\text{BK}}\breve{\Phi}^I_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}},$$

$$\text{homotopylimit}_r \text{Spec}^{\text{BK}}\Phi^r_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{homotopycolimit}_I \text{Spec}^{\text{BK}}\Phi^I_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}.$$

Definition 3.1.10. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Perfectcomplex, Condensed}_{*} \tag{3.1.55}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{CS}}\widetilde{\Delta}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Phi}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^{+}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.56}$$

$$\text{Spec}^{\text{CS}}\widetilde{\nabla}^{+}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^{\dagger}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}^{\dagger}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.57}$$

$$\tag{3.1.58}$$

$$\text{Spec}^{\text{CS}}\breve{\Delta}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\breve{\Phi}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\breve{\Delta}^{+}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.59}$$

$$\text{Spec}^{\text{CS}}\breve{\nabla}^{+}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\breve{\Delta}^{\dagger}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\breve{\nabla}^{\dagger}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.60}$$

$$\tag{3.1.61}$$

$$\text{Spec}^{\text{CS}}\Delta_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\nabla_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\Phi_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\Delta^{+}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.62}$$

$$\text{Spec}^{\text{CS}}\nabla^{+}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\Delta^{\dagger}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\nabla^{\dagger}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}. \tag{3.1.63}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^{r}_{\psi,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^{I}_{\psi,\Gamma,X},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^{r}_{\psi,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^{I}_{\psi,\Gamma,X},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^{r}_{\psi,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^{I}_{\psi,\Gamma,X}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^{r}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^{I}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^{r}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^{I}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^{r}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^{I}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}.$$

Proposition 3.1.11. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Perfectcomplex, Condensed}_{*} \tag{3.1.64}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{CS}}\widetilde{\Phi}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.65}$$

$$\tag{3.1.66}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.67}$$

$$\tag{3.1.68}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.69}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{3.1.70}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{3.1.71}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 3.1.12. Similar proposition holds for

$$\mathrm{Quasicoherentsheaves}, \mathrm{Perfectcomplex}, \mathrm{IndBanach}_*. \tag{3.1.72}$$

Definition 3.1.13. Let ψ be a toric tower over $\mathbb{Q}_p$ as in [KL2, Chapter 7] with base $\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power:

$$\widetilde{\Delta}_{\psi,\Gamma}, \widetilde{\nabla}_{\psi,\Gamma}, \widetilde{\Phi}_{\psi,\Gamma}, \widetilde{\Delta}^+_{\psi,\Gamma}, \widetilde{\nabla}^+_{\psi,\Gamma}, \widetilde{\Delta}^\dagger_{\psi,\Gamma}, \widetilde{\nabla}^\dagger_{\psi,\Gamma}, \widetilde{\Phi}^r_{\psi,\Gamma}, \widetilde{\Phi}^I_{\psi,\Gamma},$$

$$\breve{\Delta}_{\psi,\Gamma}, \breve{\nabla}_{\psi,\Gamma}, \breve{\Phi}_{\psi,\Gamma}, \breve{\Delta}^+_{\psi,\Gamma}, \breve{\nabla}^+_{\psi,\Gamma}, \breve{\Delta}^\dagger_{\psi,\Gamma}, \breve{\nabla}^\dagger_{\psi,\Gamma}, \breve{\Phi}^r_{\psi,\Gamma}, \breve{\Phi}^I_{\psi,\Gamma},$$

$$\Delta_{\psi,\Gamma}, \nabla_{\psi,\Gamma}, \Phi_{\psi,\Gamma}, \Delta^+_{\psi,\Gamma}, \nabla^+_{\psi,\Gamma}, \Delta^\dagger_{\psi,\Gamma}, \nabla^\dagger_{\psi,\Gamma}, \Phi^r_{\psi,\Gamma}, \Phi^I_{\psi,\Gamma}.$$

We now consider $\circ$ being deforming preadic space over $\mathbb{Q}_p$. Taking the product we have:

$$\widetilde{\Phi}_{\psi,\Gamma,\circ}, \widetilde{\Phi}^r_{\psi,\Gamma,\circ}, \widetilde{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\breve{\Phi}_{\psi,\Gamma,\circ}, \breve{\Phi}^r_{\psi,\Gamma,\circ}, \breve{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\Phi_{\psi,\Gamma,\circ}, \Phi^r_{\psi,\Gamma,\circ}, \Phi^I_{\psi,\Gamma,\circ}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma := \mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty,1)$-categories of $(\infty,1)$-modules.

Definition 3.1.14. First we consider the Bambozzi-Kremnizer spectrum $\mathrm{Spec}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,\circ}, \tag{3.1.73}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,\circ}, \tag{3.1.74}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,\circ}. \tag{3.1.75}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.76}$$

(3.1.77)

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.78}$$

(3.1.79)

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.80}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,\circ}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 3.1.15. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_{\psi,\Gamma}, \widetilde{\nabla}_{\psi,\Gamma}, \widetilde{\Phi}_{\psi,\Gamma}, \widetilde{\Delta}^+_{\psi,\Gamma}, \widetilde{\nabla}^+_{\psi,\Gamma}, \widetilde{\Delta}^\dagger_{\psi,\Gamma}, \widetilde{\nabla}^\dagger_{\psi,\Gamma}, \widetilde{\Phi}^r_{\psi,\Gamma}, \widetilde{\Phi}^I_{\psi,\Gamma},$$

$$\breve{\Delta}_{\psi,\Gamma}, \breve{\nabla}_{\psi,\Gamma}, \breve{\Phi}_{\psi,\Gamma}, \breve{\Delta}^+_{\psi,\Gamma}, \breve{\nabla}^+_{\psi,\Gamma}, \breve{\Delta}^\dagger_{\psi,\Gamma}, \breve{\nabla}^\dagger_{\psi,\Gamma}, \breve{\Phi}^r_{\psi,\Gamma}, \breve{\Phi}^I_{\psi,\Gamma},$$

$$\Delta_{\psi,\Gamma}, \nabla_{\psi,\Gamma}, \Phi_{\psi,\Gamma}, \Delta^+_{\psi,\Gamma}, \nabla^+_{\psi,\Gamma}, \Delta^\dagger_{\psi,\Gamma}, \nabla^\dagger_{\psi,\Gamma}, \Phi^r_{\psi,\Gamma}, \Phi^I_{\psi,\Gamma},$$

with $\circ$. Then we have the notations:

$$\widetilde{\Delta}_{\psi,\Gamma,\circ}, \widetilde{\nabla}_{\psi,\Gamma,\circ}, \widetilde{\Phi}_{\psi,\Gamma,\circ}, \widetilde{\Delta}^+_{\psi,\Gamma,\circ}, \widetilde{\nabla}^+_{\psi,\Gamma,\circ}, \widetilde{\Delta}^\dagger_{\psi,\Gamma,\circ}, \widetilde{\nabla}^\dagger_{\psi,\Gamma,\circ}, \widetilde{\Phi}^r_{\psi,\Gamma,\circ}, \widetilde{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\breve{\Delta}_{\psi,\Gamma,\circ}, \breve{\nabla}_{\psi,\Gamma,\circ}, \breve{\Phi}_{\psi,\Gamma,\circ}, \breve{\Delta}^+_{\psi,\Gamma,\circ}, \breve{\nabla}^+_{\psi,\Gamma,\circ}, \breve{\Delta}^\dagger_{\psi,\Gamma,\circ}, \breve{\nabla}^\dagger_{\psi,\Gamma,\circ}, \breve{\Phi}^r_{\psi,\Gamma,\circ}, \breve{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\Delta_{\psi,\Gamma,\circ}, \nabla_{\psi,\Gamma,\circ}, \Phi_{\psi,\Gamma,\circ}, \Delta^+_{\psi,\Gamma,\circ}, \nabla^+_{\psi,\Gamma,\circ}, \Delta^\dagger_{\psi,\Gamma,\circ}, \nabla^\dagger_{\psi,\Gamma,\circ}, \Phi^r_{\psi,\Gamma,\circ}, \Phi^I_{\psi,\Gamma,\circ}.$$

Definition 3.1.16. First we consider the Clausen-Scholze spectrum $\mathrm{Spec}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,\circ}, \tag{3.1.81}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\circ}, \tag{3.1.82}$$

$$\tag{3.1.83}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,\circ}, \breve{\nabla}_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,\circ}, \tag{3.1.84}$$
$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\circ}, \breve{\Phi}^I_{\psi,\Gamma,\circ}, \tag{3.1.85}$$
$$\tag{3.1.86}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,\circ}, \tag{3.1.87}$$
$$\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\circ}. \tag{3.1.88}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.89}$$
$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.90}$$
$$\tag{3.1.91}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.92}$$
$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.93}$$
$$\tag{3.1.94}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.95}$$
$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.96}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\circ}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 3.1.17. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves, IndBanach}_* \tag{3.1.97}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{BK}}\widetilde{\Phi}_{\psi,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.98}$$

(3.1.99)

$$\text{Spec}^{\text{BK}}\breve{\Phi}_{\psi,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.100}$$

(3.1.101)

$$\text{Spec}^{\text{BK}}\Phi_{\psi,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}. \tag{3.1.102}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,\circ}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^r_{\psi,\Gamma,\circ}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\Phi^r_{\psi,\Gamma,\circ}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\Phi^I_{\psi,\Gamma,\circ}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^r_{\psi,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^I_{\psi,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\Phi^r_{\psi,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\Phi^I_{\psi,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}.$$

Definition 3.1.18. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Condensed}_* \tag{3.1.103}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{CS}}\widetilde{\Delta}_{\psi,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}_{\psi,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Phi}_{\psi,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.104}$$

$$\text{Spec}^{\text{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.105}$$

(3.1.106)

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.107}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.108}$$

$$\tag{3.1.109}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.110}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.111}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\circ}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 3.1.19. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Condensed}_* \tag{3.1.112}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.113}$$

$$\tag{3.1.114}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.115}$$

$$\tag{3.1.116}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.117}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we

consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\circ}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\circ}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\circ}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\circ}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{3.1.118}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{3.1.119}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 3.1.20. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{IndBanach}_*. \tag{3.1.120}$$

Definition 3.1.21. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves}, \text{Perfectcomplex}, \text{IndBanach}_* \tag{3.1.121}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.122}$$

(3.1.123)

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},\tag{3.1.124}$$
$$\tag{3.1.125}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}.\tag{3.1.126}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,\circ}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 3.1.22. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\mathrm{Quasicoherentsheaves, Perfectcomplex, Condensed}_*\tag{3.1.127}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},\tag{3.1.128}$$
$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},\tag{3.1.129}$$
$$\tag{3.1.130}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},\tag{3.1.131}$$
$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},\tag{3.1.132}$$
$$\tag{3.1.133}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},\tag{3.1.134}$$
$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}.\tag{3.1.135}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\circ}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 3.1.23. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Perfectcomplex}, \mathrm{Condensed}_* \tag{3.1.136}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.137}$$

(3.1.138)

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.139}$$

(3.1.140)

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.141}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\circ}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{3.1.142}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{3.1.143}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 3.1.24. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{Perfectcomplex}, \text{IndBanach}_*. \tag{3.1.144}$$

Frobenius Quasicoherent Modules III: Deformation in $(\infty,1)$-Ind-Preadic Spaces

Definition 3.1.25. Let ψ be a toric tower over $\mathbb{Q}_p$ as in [KL2, Chapter 7] with base $\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power:

$$\widetilde{\Delta}_{\psi,\Gamma}, \widetilde{\nabla}_{\psi,\Gamma}, \widetilde{\Phi}_{\psi,\Gamma}, \widetilde{\Delta}^+_{\psi,\Gamma}, \widetilde{\nabla}^+_{\psi,\Gamma}, \widetilde{\Delta}^\dagger_{\psi,\Gamma}, \widetilde{\nabla}^\dagger_{\psi,\Gamma}, \widetilde{\Phi}^r_{\psi,\Gamma}, \widetilde{\Phi}^I_{\psi,\Gamma},$$

$$\breve{\Delta}_{\psi,\Gamma}, \breve{\nabla}_{\psi,\Gamma}, \breve{\Phi}_{\psi,\Gamma}, \breve{\Delta}^+_{\psi,\Gamma}, \breve{\nabla}^+_{\psi,\Gamma}, \breve{\Delta}^\dagger_{\psi,\Gamma}, \breve{\nabla}^\dagger_{\psi,\Gamma}, \breve{\Phi}^r_{\psi,\Gamma}, \breve{\Phi}^I_{\psi,\Gamma},$$

$$\Delta_{\psi,\Gamma}, \nabla_{\psi,\Gamma}, \Phi_{\psi,\Gamma}, \Delta^+_{\psi,\Gamma}, \nabla^+_{\psi,\Gamma}, \Delta^\dagger_{\psi,\Gamma}, \nabla^\dagger_{\psi,\Gamma}, \Phi^r_{\psi,\Gamma}, \Phi^I_{\psi,\Gamma}.$$

We now consider the following rings with $X_\square$ being a homotopy colimit

$$\underset{i}{\text{homotopycolimit}} X_{\square_i} \tag{3.1.145}$$

in the ∞-categories of analytic stacks from [BBBK] and [CS2]. Taking the product we have:

$$\widetilde{\Phi}_{\psi,\Gamma,X_\square}, \widetilde{\Phi}^r_{\psi,\Gamma,X_\square}, \widetilde{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\breve{\Phi}_{\psi,\Gamma,X_\square}, \breve{\Phi}^r_{\psi,\Gamma,X_\square}, \breve{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\Phi_{\psi,\Gamma,X_\square}, \Phi^r_{\psi,\Gamma,X_\square}, \Phi^I_{\psi,\Gamma,X_\square}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma := \mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty,1)$-categories of $(\infty,1)$-modules.

Definition 3.1.26. First we consider the Bambozzi-Kremnizer spectrum $\mathrm{Spec}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square}, \tag{3.1.146}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,X_\square}, \tag{3.1.147}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,X_\square}. \tag{3.1.148}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.149}$$

$$\tag{3.1.150}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.151}$$

$$\tag{3.1.152}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.153}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,X_\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,X_\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,X_\square}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 3.1.27. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_{\psi,\Gamma}, \widetilde{\nabla}_{\psi,\Gamma}, \widetilde{\Phi}_{\psi,\Gamma}, \widetilde{\Delta}^+_{\psi,\Gamma}, \widetilde{\nabla}^+_{\psi,\Gamma}, \widetilde{\Delta}^\dagger_{\psi,\Gamma}, \widetilde{\nabla}^\dagger_{\psi,\Gamma}, \widetilde{\Phi}^r_{\psi,\Gamma}, \widetilde{\Phi}^I_{\psi,\Gamma},$$

$$\breve{\Delta}_{\psi,\Gamma}, \breve{\nabla}_{\psi,\Gamma}, \breve{\Phi}_{\psi,\Gamma}, \breve{\Delta}^+_{\psi,\Gamma}, \breve{\nabla}^+_{\psi,\Gamma}, \breve{\Delta}^\dagger_{\psi,\Gamma}, \breve{\nabla}^\dagger_{\psi,\Gamma}, \breve{\Phi}^r_{\psi,\Gamma}, \breve{\Phi}^I_{\psi,\Gamma},$$

$$\Delta_{\psi,\Gamma}, \nabla_{\psi,\Gamma}, \Phi_{\psi,\Gamma}, \Delta^+_{\psi,\Gamma}, \nabla^+_{\psi,\Gamma}, \Delta^\dagger_{\psi,\Gamma}, \nabla^\dagger_{\psi,\Gamma}, \Phi^r_{\psi,\Gamma}, \Phi^I_{\psi,\Gamma},$$

with $X_\square$. Then we have the notations:

$$\widetilde{\Delta}_{\psi,\Gamma,X_\square}, \widetilde{\nabla}_{\psi,\Gamma,X_\square}, \widetilde{\Phi}_{\psi,\Gamma,X_\square}, \widetilde{\Delta}^+_{\psi,\Gamma,X_\square}, \widetilde{\nabla}^+_{\psi,\Gamma,X_\square}, \widetilde{\Delta}^\dagger_{\psi,\Gamma,X_\square}, \widetilde{\nabla}^\dagger_{\psi,\Gamma,X_\square}, \widetilde{\Phi}^r_{\psi,\Gamma,X_\square}, \widetilde{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\breve{\Delta}_{\psi,\Gamma,X_\square}, \breve{\nabla}_{\psi,\Gamma,X_\square}, \breve{\Phi}_{\psi,\Gamma,X_\square}, \breve{\Delta}^+_{\psi,\Gamma,X_\square}, \breve{\nabla}^+_{\psi,\Gamma,X_\square}, \breve{\Delta}^\dagger_{\psi,\Gamma,X_\square}, \breve{\nabla}^\dagger_{\psi,\Gamma,X_\square}, \breve{\Phi}^r_{\psi,\Gamma,X_\square}, \breve{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\Delta_{\psi,\Gamma,X_\square}, \nabla_{\psi,\Gamma,X_\square}, \Phi_{\psi,\Gamma,X_\square}, \Delta^+_{\psi,\Gamma,X_\square}, \nabla^+_{\psi,\Gamma,X_\square}, \Delta^\dagger_{\psi,\Gamma,X_\square}, \nabla^\dagger_{\psi,\Gamma,X_\square}, \Phi^r_{\psi,\Gamma,X_\square}, \Phi^I_{\psi,\Gamma,X_\square}.$$

Definition 3.1.28. First we consider the Clausen-Scholze spectrum $\mathrm{Spec}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,X_\square}, \tag{3.1.154}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square}, \tag{3.1.155}$$

$$\tag{3.1.156}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,X_\square}, \breve{\nabla}_{\psi,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,X_\square}, \tag{3.1.157}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X_\square}, \breve{\Phi}^I_{\psi,\Gamma,X_\square}, \tag{3.1.158}$$

$$\tag{3.1.159}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,X_\square}, \tag{3.1.160}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X_\square}. \tag{3.1.161}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.162}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.163}$$

$$\tag{3.1.164}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.165}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.166}$$

$$\tag{3.1.167}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.168}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.169}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X_\square},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X_\square},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X_\square}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 3.1.29. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM][2]:

$$\mathrm{Quasicoherentpresheaves}, \mathrm{IndBanach}_* \tag{3.1.170}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.171}$$

(3.1.172)

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.173}$$

(3.1.174)

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.175}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square},$$

[2] Here the categories are defined to be the corresponding homotopy colimits of the corresponding categories with respect to each $\square_i$.

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,X_\square}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 3.1.30. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Condensed}_* \tag{3.1.176}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.177}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.178}$$

$$\tag{3.1.179}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.180}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.181}$$

$$\tag{3.1.182}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.183}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.184}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X_\square}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 3.1.31. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Condensed}_{*} \tag{3.1.185}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{CS}}\widetilde{\Phi}_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.186}$$

$$\tag{3.1.187}$$

$$\text{Spec}^{\text{CS}}\breve{\Phi}_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.188}$$

$$\tag{3.1.189}$$

$$\text{Spec}^{\text{CS}}\Phi_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.190}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{\psi,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{\psi,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{\psi,\Gamma,X_\square}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{3.1.191}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{3.1.192}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 3.1.32. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{IndBanach}_*. \tag{3.1.193}$$

Definition 3.1.33. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves}, \text{Perfectcomplex}, \text{IndBanach}_* \tag{3.1.194}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{BK}}\widetilde{\Phi}_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.195}$$

$$\tag{3.1.196}$$

$$\text{Spec}^{\text{BK}}\breve{\Phi}_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.197}$$

$$\tag{3.1.198}$$

$$\text{Spec}^{\text{BK}}\Phi_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}. \tag{3.1.199}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^r_{\psi,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\Phi^r_{\psi,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\Phi^I_{\psi,\Gamma,X_\square}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^r_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^I_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\Phi^r_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\Phi^I_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}.$$

Definition 3.1.34. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves}, \text{Perfectcomplex}, \text{Condensed}_* \tag{3.1.200}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.201}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.202}$$

$$\tag{3.1.203}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.204}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.205}$$

$$\tag{3.1.206}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.207}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.208}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X_\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X_\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X_\square}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 3.1.35. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Perfectcomplex}, \mathrm{Condensed}_* \tag{3.1.209}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.210}$$

$$\tag{3.1.211}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.212}$$

$$\tag{3.1.213}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z}, \tag{3.1.214}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X_\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X_\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X_\square}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{3.1.215}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{3.1.216}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 3.1.36. Similar proposition holds for

$$\mathrm{Quasicoherentsheaves}, \mathrm{Perfectcomplex}, \mathrm{IndBanach}_*. \tag{3.1.217}$$

3.1.2 Univariate Hodge Iwasawa Modules

Frobenius Quasicoherent Modules I

Definition 3.1.37. Let ψ be a toric tower over $\mathbb{Q}_p$ as in [KL2, Chapter 7] with base $\mathbb{Q}_p\left\langle X_1^{\pm 1}, ..., X_k^{\pm 1}\right\rangle$. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power[3]:

$$\widetilde{\Delta}_\psi, \widetilde{\nabla}_\psi, \widetilde{\Phi}_\psi, \widetilde{\Delta}^+_\psi, \widetilde{\nabla}^+_\psi, \widetilde{\Delta}^\dagger_\psi, \widetilde{\nabla}^\dagger_\psi, \widetilde{\Phi}^r_\psi, \widetilde{\Phi}^I_\psi,$$

$$\breve{\Delta}_\psi, \breve{\nabla}_\psi, \breve{\Phi}_\psi, \breve{\Delta}^+_\psi, \breve{\nabla}^+_\psi, \breve{\Delta}^\dagger_\psi, \breve{\nabla}^\dagger_\psi, \breve{\Phi}^r_\psi, \breve{\Phi}^I_\psi,$$

$$\Delta_\psi, \nabla_\psi, \Phi_\psi, \Delta^+_\psi, \nabla^+_\psi, \Delta^\dagger_\psi, \nabla^\dagger_\psi, \Phi^r_\psi, \Phi^I_\psi.$$

Now consider X being preadic space over $\mathbb{Q}_p$. Taking the product we have:

$$\widetilde{\Phi}_{\psi,X}, \widetilde{\Phi}^r_{\psi,X}, \widetilde{\Phi}^I_{\psi,X},$$

$$\breve{\Phi}_{\psi,X}, \breve{\Phi}^r_{\psi,X}, \breve{\Phi}^I_{\psi,X},$$

$$\Phi_{\psi,X}, \Phi^r_{\psi,X}, \Phi^I_{\psi,X}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma := \mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty,1)$-categories of $(\infty,1)$-modules.

Definition 3.1.38. First we consider the Bambozzi-Kremnizer spectrum $\mathrm{Spec}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,X}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,X}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,X}, \tag{3.1.218}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,X}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,X}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,X}, \tag{3.1.219}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,X}, \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,X}, \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,X}. \tag{3.1.220}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.221}$$

(3.1.222)

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.223}$$

(3.1.224)

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.225}$$

[3]Here $|\Gamma| = 1$.

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,X}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,X},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,X}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,X},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,X}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,X}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 3.1.39. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_\psi, \widetilde{\nabla}_\psi, \widetilde{\Phi}_\psi, \widetilde{\Delta}^+_\psi, \widetilde{\nabla}^+_\psi, \widetilde{\Delta}^\dagger_\psi, \widetilde{\nabla}^\dagger_\psi, \widetilde{\Phi}^r_\psi, \widetilde{\Phi}^I_\psi,$$

$$\breve{\Delta}_\psi, \breve{\nabla}_\psi, \breve{\Phi}_\psi, \breve{\Delta}^+_\psi, \breve{\nabla}^+_\psi, \breve{\Delta}^\dagger_\psi, \breve{\nabla}^\dagger_\psi, \breve{\Phi}^r_\psi, \breve{\Phi}^I_\psi,$$

$$\Delta_\psi, \nabla_\psi, \Phi_\psi, \Delta^+_\psi, \nabla^+_\psi, \Delta^\dagger_\psi, \nabla^\dagger_\psi, \Phi^r_\psi, \Phi^I_\psi,$$

with X. Then we have the notations:

$$\widetilde{\Delta}_{\psi,X}, \widetilde{\nabla}_{\psi,X}, \widetilde{\Phi}_{\psi,X}, \widetilde{\Delta}^+_{\psi,X}, \widetilde{\nabla}^+_{\psi,X}, \widetilde{\Delta}^\dagger_{\psi,X}, \widetilde{\nabla}^\dagger_{\psi,X}, \widetilde{\Phi}^r_{\psi,X}, \widetilde{\Phi}^I_{\psi,X},$$

$$\breve{\Delta}_{\psi,X}, \breve{\nabla}_{\psi,X}, \breve{\Phi}_{\psi,X}, \breve{\Delta}^+_{\psi,X}, \breve{\nabla}^+_{\psi,X}, \breve{\Delta}^\dagger_{\psi,X}, \breve{\nabla}^\dagger_{\psi,X}, \breve{\Phi}^r_{\psi,X}, \breve{\Phi}^I_{\psi,X},$$

$$\Delta_{\psi,X}, \nabla_{\psi,X}, \Phi_{\psi,X}, \Delta^+_{\psi,X}, \nabla^+_{\psi,X}, \Delta^\dagger_{\psi,X}, \nabla^\dagger_{\psi,X}, \Phi^r_{\psi,X}, \Phi^I_{\psi,X}.$$

Definition 3.1.40. First we consider the Clausen-Scholze spectrum $\mathrm{Spec}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,X}, \tag{3.1.226}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,X}, \tag{3.1.227}$$

(3.1.228)

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,X}, \breve{\nabla}_{\psi,X}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,X}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,X}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,X}, \tag{3.1.229}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,X}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,X}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,X}, \breve{\Phi}^I_{\psi,X}, \tag{3.1.230}$$

$$\tag{3.1.231}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,X}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,X}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,X}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,X}, \mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,X}, \tag{3.1.232}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,X}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,X}, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,X}, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,X}. \tag{3.1.233}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.234}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.235}$$

$$\tag{3.1.236}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.237}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.238}$$

$$\tag{3.1.239}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.240}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.241}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,X}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 3.1.41. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves}, \text{IndBanach}_{*} \tag{3.1.242}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{BK}}\widetilde{\Phi}_{\psi,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.243}$$

(3.1.244)

$$\text{Spec}^{\text{BK}}\breve{\Phi}_{\psi,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.245}$$

(3.1.246)

$$\text{Spec}^{\text{BK}}\Phi_{\psi,X}/\text{Fro}^{\mathbb{Z}}. \tag{3.1.247}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{\psi,X}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{\psi,X},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^r_{\psi,X}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^I_{\psi,X},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\Phi^r_{\psi,X}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\Phi^I_{\psi,X}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{\psi,X}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{\psi,X}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^r_{\psi,X}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^I_{\psi,X}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\Phi^r_{\psi,X}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\Phi^I_{\psi,X}/\text{Fro}^{\mathbb{Z}}.$$

Definition 3.1.42. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves}, \text{Condensed}_{*} \tag{3.1.248}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{CS}}\widetilde{\Delta}_{\psi,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}_{\psi,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Phi}_{\psi,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^+_{\psi,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.249}$$

$$\text{Spec}^{\text{CS}}\widetilde{\nabla}^+_{\psi,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^\dagger_{\psi,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}^\dagger_{\psi,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.250}$$

(3.1.251)

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^{+}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.252}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^{+}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^{\dagger}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^{\dagger}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.253}$$

$$\tag{3.1.254}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^{+}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.255}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^{+}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^{\dagger}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^{\dagger}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.256}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{\psi,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{\psi,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{r}_{\psi,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{I}_{\psi,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{r}_{\psi,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{I}_{\psi,X}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{r}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^{I}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{r}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{I}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 3.1.43. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}_{\mathrm{Condensed}_*} \tag{3.1.257}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.258}$$

$$\tag{3.1.259}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.260}$$

$$\tag{3.1.261}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.262}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we

consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,X}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,X},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,X}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,X},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,X}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,X}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{3.1.263}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{3.1.264}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 3.1.44. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{IndBanach}_*. \tag{3.1.265}$$

Definition 3.1.45. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves}, \text{Perfectcomplex}, \text{IndBanach}_* \tag{3.1.266}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.267}$$

$$\tag{3.1.268}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.269}$$
$$\tag{3.1.270}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.271}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,X}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 3.1.46. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\mathrm{Quasicoherentsheaves, Perfectcomplex, Condensed}_{*} \tag{3.1.272}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^{+}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.273}$$
$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^{+}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^{\dagger}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^{\dagger}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.274}$$
$$\tag{3.1.275}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^{+}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.276}$$
$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^{+}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^{\dagger}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^{\dagger}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.277}$$
$$\tag{3.1.278}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^{+}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.279}$$
$$\mathrm{Spec}^{\mathrm{CS}}\nabla^{+}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^{\dagger}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^{\dagger}_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.280}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{\psi,X}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{\psi,X},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{\psi,X}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^I_{\psi,X},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{\psi,X}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{\psi,X}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{\psi,X}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{\psi,X}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{\psi,X}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{\psi,X}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{\psi,X}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{\psi,X}/\text{Fro}^{\mathbb{Z}}.$$

Proposition 3.1.47. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Perfectcomplex, Condensed}_* \tag{3.1.281}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{CS}}\widetilde{\Phi}_{\psi,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.282}$$

$$\tag{3.1.283}$$

$$\text{Spec}^{\text{CS}}\breve{\Phi}_{\psi,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.284}$$

$$\tag{3.1.285}$$

$$\text{Spec}^{\text{CS}}\Phi_{\psi,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.286}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{\psi,X}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{\psi,X},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{\psi,X}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^I_{\psi,X},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{\psi,X}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{\psi,X}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{3.1.287}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{3.1.288}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 3.1.48. Similar proposition holds for

$$\mathrm{Quasicoherentsheaves}, \mathrm{Perfectcomplex}, \mathrm{IndBanach}_{*}. \tag{3.1.289}$$

Frobenius Quasicoherent Modules II: Deformation in Preadic Spaces

Definition 3.1.49. Let ψ be a toric tower over $\mathbb{Q}_p$ as in [KL2, Chapter 7] with base $\mathbb{Q}_p\left\langle X_1^{\pm 1}, ..., X_k^{\pm 1}\right\rangle$. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power[4]:

$$\widetilde{\Delta}_\psi, \widetilde{\nabla}_\psi, \widetilde{\Phi}_\psi, \widetilde{\Delta}^+_\psi, \widetilde{\nabla}^+_\psi, \widetilde{\Delta}^\dagger_\psi, \widetilde{\nabla}^\dagger_\psi, \widetilde{\Phi}^r_\psi, \widetilde{\Phi}^I_\psi,$$

$$\breve{\Delta}_\psi, \breve{\nabla}_\psi, \breve{\Phi}_\psi, \breve{\Delta}^+_\psi, \breve{\nabla}^+_\psi, \breve{\Delta}^\dagger_\psi, \breve{\nabla}^\dagger_\psi, \breve{\Phi}^r_\psi, \breve{\Phi}^I_\psi,$$

$$\Delta_\psi, \nabla_\psi, \Phi_\psi, \Delta^+_\psi, \nabla^+_\psi, \Delta^\dagger_\psi, \nabla^\dagger_\psi, \Phi^r_\psi, \Phi^I_\psi.$$

We now consider $\circ$ being deforming preadic space over $\mathbb{Q}_p$. Taking the product we have:

$$\widetilde{\Phi}_{\psi,\circ}, \widetilde{\Phi}^r_{\psi,\circ}, \widetilde{\Phi}^I_{\psi,\circ},$$

$$\breve{\Phi}_{\psi,\circ}, \breve{\Phi}^r_{\psi,\circ}, \breve{\Phi}^I_{\psi,\circ},$$

$$\Phi_{\psi,\circ}, \Phi^r_{\psi,\circ}, \Phi^I_{\psi,\circ}.$$

[4]Here $|\Gamma| = 1$.

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma := \mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty,1)$-categories of $(\infty,1)$-modules.

Definition 3.1.50. First we consider the Bambozzi-Kremnizer spectrum $\mathrm{Spec}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\circ}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\circ}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\circ}, \tag{3.1.290}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,\circ}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\circ}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\circ}, \tag{3.1.291}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,\circ}, \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\circ}, \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\circ}. \tag{3.1.292}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.293}$$

$$\tag{3.1.294}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.295}$$

$$\tag{3.1.296}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.297}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\circ}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 3.1.51. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_\psi, \widetilde{\nabla}_\psi, \widetilde{\Phi}_\psi, \widetilde{\Delta}^+_\psi, \widetilde{\nabla}^+_\psi, \widetilde{\Delta}^\dagger_\psi, \widetilde{\nabla}^\dagger_\psi, \widetilde{\Phi}^r_\psi, \widetilde{\Phi}^I_\psi,$$

$$\breve{\Delta}_\psi, \breve{\nabla}_\psi, \breve{\Phi}_\psi, \breve{\Delta}^+_\psi, \breve{\nabla}^+_\psi, \breve{\Delta}^\dagger_\psi, \breve{\nabla}^\dagger_\psi, \breve{\Phi}^r_\psi, \breve{\Phi}^I_\psi,$$

$$\Delta_\psi, \nabla_\psi, \Phi_\psi, \Delta^+_\psi, \nabla^+_\psi, \Delta^\dagger_\psi, \nabla^\dagger_\psi, \Phi^r_\psi, \Phi^I_\psi,$$

with $\circ$. Then we have the notations:

$$\widetilde{\Delta}_{\psi,\circ}, \widetilde{\nabla}_{\psi,\circ}, \widetilde{\Phi}_{\psi,\circ}, \widetilde{\Delta}^+_{\psi,\circ}, \widetilde{\nabla}^+_{\psi,\circ}, \widetilde{\Delta}^\dagger_{\psi,\circ}, \widetilde{\nabla}^\dagger_{\psi,\circ}, \widetilde{\Phi}^r_{\psi,\circ}, \widetilde{\Phi}^I_{\psi,\circ},$$

$$\breve{\Delta}_{\psi,\circ}, \breve{\nabla}_{\psi,\circ}, \breve{\Phi}_{\psi,\circ}, \breve{\Delta}^+_{\psi,\circ}, \breve{\nabla}^+_{\psi,\circ}, \breve{\Delta}^\dagger_{\psi,\circ}, \breve{\nabla}^\dagger_{\psi,\circ}, \breve{\Phi}^r_{\psi,\circ}, \breve{\Phi}^I_{\psi,\circ},$$

$$\Delta_{\psi,\circ}, \nabla_{\psi,\circ}, \Phi_{\psi,\circ}, \Delta^+_{\psi,\circ}, \nabla^+_{\psi,\circ}, \Delta^\dagger_{\psi,\circ}, \nabla^\dagger_{\psi,\circ}, \Phi^r_{\psi,\circ}, \Phi^I_{\psi,\circ}.$$

Definition 3.1.52. First we consider the Clausen-Scholze spectrum $\mathrm{Spec}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\circ}, \tag{3.1.298}$$
$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\circ}, \tag{3.1.299}$$
$$\tag{3.1.300}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\circ}, \breve{\nabla}_{\psi,\circ}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\circ}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\circ}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\circ}, \tag{3.1.301}$$
$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\circ}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\circ}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\circ}, \breve{\Phi}^I_{\psi,\circ}, \tag{3.1.302}$$
$$\tag{3.1.303}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\circ}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\circ}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\circ}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\circ}, \mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\circ}, \tag{3.1.304}$$
$$\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\circ}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\circ}, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\circ}, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\circ}. \tag{3.1.305}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.306}$$
$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.307}$$
$$\tag{3.1.308}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.309}$$
$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.310}$$
$$\tag{3.1.311}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.312}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.313}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\circ}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 3.1.53. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\mathrm{Quasicoherentpresheaves}, \mathrm{IndBanach}_* \tag{3.1.314}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.315}$$

(3.1.316)

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.317}$$

(3.1.318)

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.319}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\circ},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\circ}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\circ}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 3.1.54. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves}, \text{Condensed}_* \tag{3.1.320}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.321}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.322}$$

$$\tag{3.1.323}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.324}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.325}$$

$$\tag{3.1.326}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.327}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.328}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\circ}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\circ},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\circ}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\circ},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\circ}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\circ}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 3.1.55. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Condensed}_* \tag{3.1.329}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{CS}}\widetilde{\Phi}_{\psi,\circ}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.330}$$

$$\tag{3.1.331}$$

$$\text{Spec}^{\text{CS}}\breve{\Phi}_{\psi,\circ}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.332}$$

$$\tag{3.1.333}$$

$$\text{Spec}^{\text{CS}}\Phi_{\psi,\circ}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.334}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\circ}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\circ},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{\psi,\circ}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^I_{\psi,\circ},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{\psi,\circ}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{\psi,\circ}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\circ}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\circ}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{\psi,\circ}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\circ}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{\psi,\circ}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{\psi,\circ}/\text{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{3.1.335}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{3.1.336}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 3.1.56. Similar proposition holds for

$$\mathrm{Quasicoherentsheaves}, \mathrm{IndBanach}_*. \tag{3.1.337}$$

Definition 3.1.57. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Perfectcomplex}, \mathrm{IndBanach}_* \tag{3.1.338}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.339}$$

$$\tag{3.1.340}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.341}$$

$$\tag{3.1.342}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.343}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\circ}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 3.1.58. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\mathrm{Quasicoherentsheaves}, \mathrm{Perfectcomplex}, \mathrm{Condensed}_* \tag{3.1.344}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^{+}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.345}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^{+}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^{\dagger}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^{\dagger}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.346}$$

$$\tag{3.1.347}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^{+}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.348}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^{+}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^{\dagger}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^{\dagger}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.349}$$

$$\tag{3.1.350}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^{+}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.351}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^{+}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^{\dagger}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^{\dagger}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.352}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{\psi,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{\psi,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{r}_{\psi,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{I}_{\psi,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{r}_{\psi,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{I}_{\psi,\circ}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{r}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^{I}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{r}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{I}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 3.1.59. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Perfectcomplex}, \mathrm{Condensed}_{*} \tag{3.1.353}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.354}$$

$$\tag{3.1.355}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.356}$$

$$\tag{3.1.357}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.358}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\circ}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{3.1.359}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{3.1.360}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 3.1.60. Similar proposition holds for

$$\mathrm{Quasicoherentsheaves}, \mathrm{Perfectcomplex}, \mathrm{IndBanach}_*. \tag{3.1.361}$$

3.1.3 Multivariate Hodge Iwasawa Prestacks

Frobenius Quasicoherent Prestacks I

Definition 3.1.61. We now consider the pro-étale site of $\mathrm{Spa}\mathbb{Q}_p\left\langle X_1^{\pm 1}, ..., X_k^{\pm 1}\right\rangle$ from [Sch], denote that by $*$. To be more accurate we replace one component for Γ with the pro-étale site of $\mathrm{Spa}\mathbb{Q}_p\left\langle X_1^{\pm 1}, ..., X_k^{\pm 1}\right\rangle$.

And we treat then all the functor to be prestacks for this site[5]. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power:

$$\widetilde{\Delta}_{*,\Gamma}, \widetilde{\nabla}_{*,\Gamma}, \widetilde{\Phi}_{*,\Gamma}, \widetilde{\Delta}^+_{*,\Gamma}, \widetilde{\nabla}^+_{*,\Gamma}, \widetilde{\Delta}^\dagger_{*,\Gamma}, \widetilde{\nabla}^\dagger_{*,\Gamma}, \widetilde{\Phi}^r_{*,\Gamma}, \widetilde{\Phi}^I_{*,\Gamma},$$

$$\breve{\Delta}_{*,\Gamma}, \breve{\nabla}_{*,\Gamma}, \breve{\Phi}_{*,\Gamma}, \breve{\Delta}^+_{*,\Gamma}, \breve{\nabla}^+_{*,\Gamma}, \breve{\Delta}^\dagger_{*,\Gamma}, \breve{\nabla}^\dagger_{*,\Gamma}, \breve{\Phi}^r_{*,\Gamma}, \breve{\Phi}^I_{*,\Gamma},$$

$$\Delta_{*,\Gamma}, \nabla_{*,\Gamma}, \Phi_{*,\Gamma}, \Delta^+_{*,\Gamma}, \nabla^+_{*,\Gamma}, \Delta^\dagger_{*,\Gamma}, \nabla^\dagger_{*,\Gamma}, \Phi^r_{*,\Gamma}, \Phi^I_{*,\Gamma}.$$

Taking the product we have:

$$\widetilde{\Phi}_{*,\Gamma,X}, \widetilde{\Phi}^r_{*,\Gamma,X}, \widetilde{\Phi}^I_{*,\Gamma,X},$$

$$\breve{\Phi}_{*,\Gamma,X}, \breve{\Phi}^r_{*,\Gamma,X}, \breve{\Phi}^I_{*,\Gamma,X},$$

$$\Phi_{*,\Gamma,X}, \Phi^r_{*,\Gamma,X}, \Phi^I_{*,\Gamma,X}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma := \mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty,1)$-categories of $(\infty,1)$-modules.

Definition 3.1.62. First we consider the Bambozzi-Kremnizer spectrum $\mathrm{Spec}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,\Gamma,X}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,X}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,X}, \tag{3.1.362}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,\Gamma,X}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,X}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,X}, \tag{3.1.363}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,\Gamma,X}, \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,X}, \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,X}. \tag{3.1.364}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.365}$$

(3.1.366)

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.367}$$

(3.1.368)

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.369}$$

[5]Here for those imperfect rings, the notation will mean that the specific component forming the pro-étale site will be the perfect version of the corresponding ring. Certainly if we have $|\Gamma| = 1$ then we have that all the rings are perfect in [KL1] and [KL2].

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,X},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,X},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,X}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 3.1.63. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_{*,\Gamma}, \widetilde{\nabla}_{*,\Gamma}, \widetilde{\Phi}_{*,\Gamma}, \widetilde{\Delta}^+_{*,\Gamma}, \widetilde{\nabla}^+_{*,\Gamma}, \widetilde{\Delta}^\dagger_{*,\Gamma}, \widetilde{\nabla}^\dagger_{*,\Gamma}, \widetilde{\Phi}^r_{*,\Gamma}, \widetilde{\Phi}^I_{*,\Gamma},$$

$$\breve{\Delta}_{*,\Gamma}, \breve{\nabla}_{*,\Gamma}, \breve{\Phi}_{*,\Gamma}, \breve{\Delta}^+_{*,\Gamma}, \breve{\nabla}^+_{*,\Gamma}, \breve{\Delta}^\dagger_{*,\Gamma}, \breve{\nabla}^\dagger_{*,\Gamma}, \breve{\Phi}^r_{*,\Gamma}, \breve{\Phi}^I_{*,\Gamma},$$

$$\Delta_{*,\Gamma}, \nabla_{*,\Gamma}, \Phi_{*,\Gamma}, \Delta^+_{*,\Gamma}, \nabla^+_{*,\Gamma}, \Delta^\dagger_{*,\Gamma}, \nabla^\dagger_{*,\Gamma}, \Phi^r_{*,\Gamma}, \Phi^I_{*,\Gamma},$$

with X. Then we have the notations:

$$\widetilde{\Delta}_{*,\Gamma,X}, \widetilde{\nabla}_{*,\Gamma,X}, \widetilde{\Phi}_{*,\Gamma,X}, \widetilde{\Delta}^+_{*,\Gamma,X}, \widetilde{\nabla}^+_{*,\Gamma,X}, \widetilde{\Delta}^\dagger_{*,\Gamma,X}, \widetilde{\nabla}^\dagger_{*,\Gamma,X}, \widetilde{\Phi}^r_{*,\Gamma,X}, \widetilde{\Phi}^I_{*,\Gamma,X},$$

$$\breve{\Delta}_{*,\Gamma,X}, \breve{\nabla}_{*,\Gamma,X}, \breve{\Phi}_{*,\Gamma,X}, \breve{\Delta}^+_{*,\Gamma,X}, \breve{\nabla}^+_{*,\Gamma,X}, \breve{\Delta}^\dagger_{*,\Gamma,X}, \breve{\nabla}^\dagger_{*,\Gamma,X}, \breve{\Phi}^r_{*,\Gamma,X}, \breve{\Phi}^I_{*,\Gamma,X},$$

$$\Delta_{*,\Gamma,X}, \nabla_{*,\Gamma,X}, \Phi_{*,\Gamma,X}, \Delta^+_{*,\Gamma,X}, \nabla^+_{*,\Gamma,X}, \Delta^\dagger_{*,\Gamma,X}, \nabla^\dagger_{*,\Gamma,X}, \Phi^r_{*,\Gamma,X}, \Phi^I_{*,\Gamma,X}.$$

Definition 3.1.64. First we consider the Clausen-Scholze spectrum $\mathrm{Spec}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,\Gamma,X}, \tag{3.1.370}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,X}, \tag{3.1.371}$$

$$\tag{3.1.372}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,X}, \breve{\nabla}_{*,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,\Gamma,X}, \tag{3.1.373}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,X}, \breve{\Phi}^I_{*,\Gamma,X}, \tag{3.1.374}$$

$$\tag{3.1.375}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,\Gamma,X}, \tag{3.1.376}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,X}. \tag{3.1.377}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.378}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.379}$$

$$\tag{3.1.380}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.381}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.382}$$

$$\tag{3.1.383}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.384}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.385}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,\Gamma,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,X}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 3.1.65. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves, IndBanach}_{*} \tag{3.1.386}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{BK}}\widetilde{\Phi}_{*,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.387}$$

(3.1.388)

$$\text{Spec}^{\text{BK}}\breve{\Phi}_{*,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.389}$$

(3.1.390)

$$\text{Spec}^{\text{BK}}\Phi_{*,\Gamma,X}/\text{Fro}^{\mathbb{Z}}. \tag{3.1.391}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{*,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{*,\Gamma,X},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^r_{*,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^I_{*,\Gamma,X},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\Phi^r_{*,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\Phi^I_{*,\Gamma,X}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{*,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{*,\Gamma,X}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^r_{*,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^I_{*,\Gamma,X}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\Phi^r_{*,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\Phi^I_{*,\Gamma,X}/\text{Fro}^{\mathbb{Z}}.$$

Definition 3.1.66. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Condensed}_{*} \tag{3.1.392}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{CS}}\widetilde{\Delta}_{*,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}_{*,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Phi}_{*,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^+_{*,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.393}$$

$$\text{Spec}^{\text{CS}}\widetilde{\nabla}^+_{*,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^\dagger_{*,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}^\dagger_{*,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.394}$$

(3.1.395)

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^{+}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.396}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^{+}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^{\dagger}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^{\dagger}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.397}$$

$$\tag{3.1.398}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^{+}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.399}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^{+}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^{\dagger}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^{\dagger}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.400}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,\Gamma,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,\Gamma,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,\Gamma,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{I}_{*,\Gamma,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{r}_{*,\Gamma,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{I}_{*,\Gamma,X}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^{I}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{r}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{I}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 3.1.67. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Condensed}_{*} \tag{3.1.401}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.402}$$

$$\tag{3.1.403}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.404}$$

$$\tag{3.1.405}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.406}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we

consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,X},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,\Gamma,X},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,X}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{3.1.407}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{3.1.408}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 3.1.68. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{IndBanach}_*. \tag{3.1.409}$$

Definition 3.1.69. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves}, \text{Perfectcomplex}, \text{IndBanach}_* \tag{3.1.410}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.411}$$

$$\tag{3.1.412}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.413}$$

$$\tag{3.1.414}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.415}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,X}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 3.1.70. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\mathrm{Quasicoherentsheaves, Perfectcomplex, Condensed}_{*} \tag{3.1.416}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^{+}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.417}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^{+}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^{\dagger}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^{\dagger}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.418}$$

$$\tag{3.1.419}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^{+}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.420}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^{+}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^{\dagger}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^{\dagger}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.421}$$

$$\tag{3.1.422}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^{+}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.423}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^{+}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^{\dagger}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^{\dagger}_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.424}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{*,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{*,\Gamma,X},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{*,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^I_{*,\Gamma,X},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{*,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{*,\Gamma,X}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{*,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{*,\Gamma,X}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{*,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{*,\Gamma,X}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{*,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{*,\Gamma,X}/\text{Fro}^{\mathbb{Z}}.$$

Proposition 3.1.71. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Perfectcomplex, Condensed}_* \tag{3.1.425}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{CS}}\widetilde{\Phi}_{*,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.426}$$

(3.1.427)

$$\text{Spec}^{\text{CS}}\breve{\Phi}_{*,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.428}$$

(3.1.429)

$$\text{Spec}^{\text{CS}}\Phi_{*,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.430}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{*,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{*,\Gamma,X},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{*,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^I_{*,\Gamma,X},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{*,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{*,\Gamma,X}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{3.1.431}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{3.1.432}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 3.1.72. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{Perfectcomplex}, \text{IndBanach}_*. \tag{3.1.433}$$

Frobenius Quasicoherent Prestacks II: Deformation in Preadic Spaces

Definition 3.1.73. We now consider the pro-étale site of $\mathrm{Spa}\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$, denote that by $*$. To be more accurate we replace one component for Γ with the pro-étale site of $\mathrm{Spa}\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$. And we treat then all the functor to be prestacks for this site. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power:

$$\widetilde{\Delta}_{*,\Gamma}, \widetilde{\nabla}_{*,\Gamma}, \widetilde{\Phi}_{*,\Gamma}, \widetilde{\Delta}^+_{*,\Gamma}, \widetilde{\nabla}^+_{*,\Gamma}, \widetilde{\Delta}^\dagger_{*,\Gamma}, \widetilde{\nabla}^\dagger_{*,\Gamma}, \widetilde{\Phi}^r_{*,\Gamma}, \widetilde{\Phi}^I_{*,\Gamma},$$

$$\breve{\Delta}_{*,\Gamma}, \breve{\nabla}_{*,\Gamma}, \breve{\Phi}_{*,\Gamma}, \breve{\Delta}^+_{*,\Gamma}, \breve{\nabla}^+_{*,\Gamma}, \breve{\Delta}^\dagger_{*,\Gamma}, \breve{\nabla}^\dagger_{*,\Gamma}, \breve{\Phi}^r_{*,\Gamma}, \breve{\Phi}^I_{*,\Gamma},$$

$$\Delta_{*,\Gamma}, \nabla_{*,\Gamma}, \Phi_{*,\Gamma}, \Delta^+_{*,\Gamma}, \nabla^+_{*,\Gamma}, \Delta^\dagger_{*,\Gamma}, \nabla^\dagger_{*,\Gamma}, \Phi^r_{*,\Gamma}, \Phi^I_{*,\Gamma}.$$

Taking the product we have:

$$\widetilde{\Phi}_{*,\Gamma,\circ}, \widetilde{\Phi}^r_{*,\Gamma,\circ}, \widetilde{\Phi}^I_{*,\Gamma,\circ},$$

$$\breve{\Phi}_{*,\Gamma,\circ}, \breve{\Phi}^r_{*,\Gamma,\circ}, \breve{\Phi}^I_{*,\Gamma,\circ},$$

$$\Phi_{*,\Gamma,\circ}, \Phi^r_{*,\Gamma,\circ}, \Phi^I_{*,\Gamma,\circ}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma := \mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty,1)$-categories of $(\infty,1)$-modules.

Definition 3.1.74. First we consider the Bambozzi-Kremnizer spectrum $\mathrm{Spec}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,\circ}, \tag{3.1.434}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,\circ}, \tag{3.1.435}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,\circ}. \tag{3.1.436}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.437}$$

(3.1.438)

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.439}$$

(3.1.440)

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.441}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,\circ}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 3.1.75. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_{*,\Gamma}, \widetilde{\nabla}_{*,\Gamma}, \widetilde{\Phi}_{*,\Gamma}, \widetilde{\Delta}^+_{*,\Gamma}, \widetilde{\nabla}^+_{*,\Gamma}, \widetilde{\Delta}^\dagger_{*,\Gamma}, \widetilde{\nabla}^\dagger_{*,\Gamma}, \widetilde{\Phi}^r_{*,\Gamma}, \widetilde{\Phi}^I_{*,\Gamma},$$

$$\breve{\Delta}_{*,\Gamma}, \breve{\nabla}_{*,\Gamma}, \breve{\Phi}_{*,\Gamma}, \breve{\Delta}^+_{*,\Gamma}, \breve{\nabla}^+_{*,\Gamma}, \breve{\Delta}^\dagger_{*,\Gamma}, \breve{\nabla}^\dagger_{*,\Gamma}, \breve{\Phi}^r_{*,\Gamma}, \breve{\Phi}^I_{*,\Gamma},$$

$$\Delta_{*,\Gamma}, \nabla_{*,\Gamma}, \Phi_{*,\Gamma}, \Delta^+_{*,\Gamma}, \nabla^+_{*,\Gamma}, \Delta^\dagger_{*,\Gamma}, \nabla^\dagger_{*,\Gamma}, \Phi^r_{*,\Gamma}, \Phi^I_{*,\Gamma},$$

with $\circ$. Then we have the notations:

$$\widetilde{\Delta}_{*,\Gamma,\circ}, \widetilde{\nabla}_{*,\Gamma,\circ}, \widetilde{\Phi}_{*,\Gamma,\circ}, \widetilde{\Delta}^+_{*,\Gamma,\circ}, \widetilde{\nabla}^+_{*,\Gamma,\circ}, \widetilde{\Delta}^\dagger_{*,\Gamma,\circ}, \widetilde{\nabla}^\dagger_{*,\Gamma,\circ}, \widetilde{\Phi}^r_{*,\Gamma,\circ}, \widetilde{\Phi}^I_{*,\Gamma,\circ},$$

$$\breve{\Delta}_{*,\Gamma,\circ}, \breve{\nabla}_{*,\Gamma,\circ}, \breve{\Phi}_{*,\Gamma,\circ}, \breve{\Delta}^+_{*,\Gamma,\circ}, \breve{\nabla}^+_{*,\Gamma,\circ}, \breve{\Delta}^\dagger_{*,\Gamma,\circ}, \breve{\nabla}^\dagger_{*,\Gamma,\circ}, \breve{\Phi}^r_{*,\Gamma,\circ}, \breve{\Phi}^I_{*,\Gamma,\circ},$$

$$\Delta_{*,\Gamma,\circ}, \nabla_{*,\Gamma,\circ}, \Phi_{*,\Gamma,\circ}, \Delta^+_{*,\Gamma,\circ}, \nabla^+_{*,\Gamma,\circ}, \Delta^\dagger_{*,\Gamma,\circ}, \nabla^\dagger_{*,\Gamma,\circ}, \Phi^r_{*,\Gamma,\circ}, \Phi^I_{*,\Gamma,\circ}.$$

Definition 3.1.76. First we consider the Clausen-Scholze spectrum $\mathrm{Spec}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,\Gamma,\circ}, \tag{3.1.442}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,\circ}, \tag{3.1.443}$$

$$\tag{3.1.444}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,\circ}, \breve{\nabla}_{*,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,\Gamma,\circ}, \tag{3.1.445}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,\circ}, \breve{\Phi}^I_{*,\Gamma,\circ}, \tag{3.1.446}$$

$$\tag{3.1.447}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,\Gamma,\circ}, \tag{3.1.448}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,\circ}. \tag{3.1.449}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.450}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.451}$$

$$\tag{3.1.452}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.453}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.454}$$

$$\tag{3.1.455}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^{+}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.456}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^{+}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^{\dagger}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^{\dagger}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.457}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{I}_{*,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{r}_{*,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{I}_{*,\Gamma,\circ}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^{I}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{r}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{I}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 3.1.77. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\mathrm{Quasicoherentpresheaves}, \mathrm{IndBanach}_{*} \tag{3.1.458}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.459}$$

$$\tag{3.1.460}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.461}$$

$$\tag{3.1.462}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.463}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^{r}_{*,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^{I}_{*,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^{r}_{*,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^{I}_{*,\Gamma,\circ},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,\circ}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,\circ}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 3.1.78. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves}, \text{Condensed}_* \tag{3.1.464}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.465}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.466}$$

$$\tag{3.1.467}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.468}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.469}$$

$$\tag{3.1.470}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.471}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.472}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,\circ}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,\circ},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,\circ}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,\Gamma,\circ},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,\circ}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,\circ}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 3.1.79. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Condensed}_{*} \tag{3.1.473}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{CS}}\widetilde{\Phi}_{*,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.474}$$

$$\tag{3.1.475}$$

$$\text{Spec}^{\text{CS}}\breve{\Phi}_{*,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.476}$$

$$\tag{3.1.477}$$

$$\text{Spec}^{\text{CS}}\Phi_{*,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.478}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^{r}_{*,\Gamma,\circ}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^{I}_{*,\Gamma,\circ},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^{r}_{*,\Gamma,\circ}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^{I}_{*,\Gamma,\circ},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^{r}_{*,\Gamma,\circ}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^{I}_{*,\Gamma,\circ}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^{r}_{*,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^{I}_{*,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^{r}_{*,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^{I}_{*,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^{r}_{*,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^{I}_{*,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{3.1.479}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{3.1.480}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 3.1.80. Similar proposition holds for

$$\text{Quasicoherentsheaves, IndBanach}_{*}. \tag{3.1.481}$$

Definition 3.1.81. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves, Perfectcomplex, IndBanach}_{*} \tag{3.1.482}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{BK}}\widetilde{\Phi}_{*,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.483}$$

(3.1.484)

$$\text{Spec}^{\text{BK}}\breve{\Phi}_{*,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.485}$$

(3.1.486)

$$\text{Spec}^{\text{BK}}\Phi_{*,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}. \tag{3.1.487}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^{r}_{*,\Gamma,\circ}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^{I}_{*,\Gamma,\circ},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^{r}_{*,\Gamma,\circ}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^{I}_{*,\Gamma,\circ},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\Phi^{r}_{*,\Gamma,\circ}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\Phi^{I}_{*,\Gamma,\circ}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^{r}_{*,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^{I}_{*,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^{r}_{*,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^{I}_{*,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\Phi^{r}_{*,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\Phi^{I}_{*,\Gamma,\circ}/\text{Fro}^{\mathbb{Z}}.$$

Definition 3.1.82. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Perfectcomplex, Condensed}_{*} \tag{3.1.488}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.489}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.490}$$

$$\tag{3.1.491}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.492}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.493}$$

$$\tag{3.1.494}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.495}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.496}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,\circ}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 3.1.83. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Perfectcomplex}, \mathrm{Condensed}_* \tag{3.1.497}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.498}$$

$$\tag{3.1.499}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.500}$$

$$\tag{3.1.501}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.502}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,\circ}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{3.1.503}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{3.1.504}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 3.1.84. Similar proposition holds for

$$\mathrm{Quasicoherentsheaves}, \mathrm{Perfectcomplex}, \mathrm{IndBanach}_{*}. \tag{3.1.505}$$

Frobenius Quasicoherent Prestacks III: Deformation in $(\infty, 1)$-Ind-Preadic Spaces

Definition 3.1.85. We now consider the pro-étale site of $\mathrm{Spa}\mathbb{Q}_p\left\langle X_1^{\pm 1}, ..., X_k^{\pm 1}\right\rangle$, denote that by $*$. To be more accurate we replace one component for Γ with the pro-étale site of $\mathrm{Spa}\mathbb{Q}_p\left\langle X_1^{\pm 1}, ..., X_k^{\pm 1}\right\rangle$. And we treat then all the functor to be prestacks for this site. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power:

$$\widetilde{\Delta}_{*,\Gamma},\widetilde{\nabla}_{*,\Gamma},\widetilde{\Phi}_{*,\Gamma},\widetilde{\Delta}^+_{*,\Gamma},\widetilde{\nabla}^+_{*,\Gamma},\widetilde{\Delta}^\dagger_{*,\Gamma},\widetilde{\nabla}^\dagger_{*,\Gamma},\widetilde{\Phi}^r_{*,\Gamma},\widetilde{\Phi}^I_{*,\Gamma},$$

$$\breve{\Delta}_{*,\Gamma},\breve{\nabla}_{*,\Gamma},\breve{\Phi}_{*,\Gamma},\breve{\Delta}^+_{*,\Gamma},\breve{\nabla}^+_{*,\Gamma},\breve{\Delta}^\dagger_{*,\Gamma},\breve{\nabla}^\dagger_{*,\Gamma},\breve{\Phi}^r_{*,\Gamma},\breve{\Phi}^I_{*,\Gamma},$$

$$\Delta_{*,\Gamma},\nabla_{*,\Gamma},\Phi_{*,\Gamma},\Delta^+_{*,\Gamma},\nabla^+_{*,\Gamma},\Delta^\dagger_{*,\Gamma},\nabla^\dagger_{*,\Gamma},\Phi^r_{*,\Gamma},\Phi^I_{*,\Gamma}.$$

Taking the product we have:

$$\widetilde{\Phi}_{*,\Gamma,X_\square},\widetilde{\Phi}^r_{*,\Gamma,X_\square},\widetilde{\Phi}^I_{*,\Gamma,X_\square},$$

$$\breve{\Phi}_{*,\Gamma,X_\square},\breve{\Phi}^r_{*,\Gamma,X_\square},\breve{\Phi}^I_{*,\Gamma,X_\square},$$

$$\Phi_{*,\Gamma,X_\square},\Phi^r_{*,\Gamma,X_\square},\Phi^I_{*,\Gamma,X_\square}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma := \mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty,1)$-categories of $(\infty,1)$-modules.

Definition 3.1.86. First we consider the Bambozzi-Kremnizer spectrum $\mathrm{Spec}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,\Gamma,X_\square},\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,X_\square},\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,X_\square}, \tag{3.1.506}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,\Gamma,X_\square},\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,X_\square},\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,X_\square}, \tag{3.1.507}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,\Gamma,X_\square},\mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,X_\square},\mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,X_\square}. \tag{3.1.508}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.509}$$

$$\tag{3.1.510}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.511}$$

$$\tag{3.1.512}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.513}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r,\bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,X_\square},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,X_\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,X_\square},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,X_\square},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,X_\square},\ \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,X_\square}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\ \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\ \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\ \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 3.1.87. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_{*,\Gamma}, \widetilde{\nabla}_{*,\Gamma}, \widetilde{\Phi}_{*,\Gamma}, \widetilde{\Delta}^+_{*,\Gamma}, \widetilde{\nabla}^+_{*,\Gamma}, \widetilde{\Delta}^\dagger_{*,\Gamma}, \widetilde{\nabla}^\dagger_{*,\Gamma}, \widetilde{\Phi}^r_{*,\Gamma}, \widetilde{\Phi}^I_{*,\Gamma},$$

$$\breve{\Delta}_{*,\Gamma}, \breve{\nabla}_{*,\Gamma}, \breve{\Phi}_{*,\Gamma}, \breve{\Delta}^+_{*,\Gamma}, \breve{\nabla}^+_{*,\Gamma}, \breve{\Delta}^\dagger_{*,\Gamma}, \breve{\nabla}^\dagger_{*,\Gamma}, \breve{\Phi}^r_{*,\Gamma}, \breve{\Phi}^I_{*,\Gamma},$$

$$\Delta_{*,\Gamma}, \nabla_{*,\Gamma}, \Phi_{*,\Gamma}, \Delta^+_{*,\Gamma}, \nabla^+_{*,\Gamma}, \Delta^\dagger_{*,\Gamma}, \nabla^\dagger_{*,\Gamma}, \Phi^r_{*,\Gamma}, \Phi^I_{*,\Gamma},$$

with $X_\square$. Then we have the notations:

$$\widetilde{\Delta}_{*,\Gamma,X_\square}, \widetilde{\nabla}_{*,\Gamma,X_\square}, \widetilde{\Phi}_{*,\Gamma,X_\square}, \widetilde{\Delta}^+_{*,\Gamma,X_\square}, \widetilde{\nabla}^+_{*,\Gamma,X_\square}, \widetilde{\Delta}^\dagger_{*,\Gamma,X_\square}, \widetilde{\nabla}^\dagger_{*,\Gamma,X_\square}, \widetilde{\Phi}^r_{*,\Gamma,X_\square}, \widetilde{\Phi}^I_{*,\Gamma,X_\square},$$

$$\breve{\Delta}_{*,\Gamma,X_\square}, \breve{\nabla}_{*,\Gamma,X_\square}, \breve{\Phi}_{*,\Gamma,X_\square}, \breve{\Delta}^+_{*,\Gamma,X_\square}, \breve{\nabla}^+_{*,\Gamma,X_\square}, \breve{\Delta}^\dagger_{*,\Gamma,X_\square}, \breve{\nabla}^\dagger_{*,\Gamma,X_\square}, \breve{\Phi}^r_{*,\Gamma,X_\square}, \breve{\Phi}^I_{*,\Gamma,X_\square},$$

$$\Delta_{*,\Gamma,X_\square}, \nabla_{*,\Gamma,X_\square}, \Phi_{*,\Gamma,X_\square}, \Delta^+_{*,\Gamma,X_\square}, \nabla^+_{*,\Gamma,X_\square}, \Delta^\dagger_{*,\Gamma,X_\square}, \nabla^\dagger_{*,\Gamma,X_\square}, \Phi^r_{*,\Gamma,X_\square}, \Phi^I_{*,\Gamma,X_\square}.$$

Definition 3.1.88. First we consider the Clausen-Scholze spectrum $\mathrm{Spec}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,\Gamma,X_\square}, \tag{3.1.514}$$
$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,X_\square}, \tag{3.1.515}$$
$$\tag{3.1.516}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,X_\square}, \breve{\nabla}_{*,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,\Gamma,X_\square}, \tag{3.1.517}$$
$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,X_\square}, \breve{\Phi}^I_{*,\Gamma,X_\square}, \tag{3.1.518}$$
$$\tag{3.1.519}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,\Gamma,X_\square}, \tag{3.1.520}$$
$$\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,X_\square}, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,X_\square}. \tag{3.1.521}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.522}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.523}$$

$$\tag{3.1.524}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\breve{\nabla}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.525}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.526}$$

$$\tag{3.1.527}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.528}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.529}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r,\bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,X_\square},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,X_\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,X_\square},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,\Gamma,X_\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,X_\square},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,X_\square}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 3.1.89. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\mathrm{Quasicoherentpresheaves},\mathrm{IndBanach}_* \tag{3.1.530}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.531}$$

$$\tag{3.1.532}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.533}$$

$$\tag{3.1.534}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.535}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,X_\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,X_\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,X_\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,X_\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,X_\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,X_\square}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 3.1.90. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\mathrm{Quasicoherentsheaves}, \mathrm{Condensed}_* \tag{3.1.536}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.537}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.538}$$

$$\tag{3.1.539}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.540}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.541}$$

$$\tag{3.1.542}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.543}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.544}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,X_\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,X_\square},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,\Gamma,X_\square},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,X_\square}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 3.1.91. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Condensed}_* \tag{3.1.545}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.546}$$

(3.1.547)

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.548}$$

(3.1.549)

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.550}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,X_\square},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,\Gamma,X_\square},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,X_\square}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{3.1.551}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{3.1.552}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 3.1.92. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{IndBanach}_*. \tag{3.1.553}$$

Definition 3.1.93. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves}, \text{Perfectcomplex}, \text{IndBanach}_* \tag{3.1.554}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.555}$$

$$\tag{3.1.556}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.557}$$

$$\tag{3.1.558}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.559}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,X_\square},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,X_\square},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,X_\square}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 3.1.94. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Perfectcomplex, Condensed}_* \tag{3.1.560}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.561}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.562}$$

$$\tag{3.1.563}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.564}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.565}$$

$$\tag{3.1.566}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.567}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.568}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,X_\square},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,\Gamma,X_\square},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,X_\square}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 3.1.95. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Perfectcomplex, Condensed}_* \tag{3.1.569}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.570}$$

$$\tag{3.1.571}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.572}$$

$$\tag{3.1.573}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.574}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,X_\square},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,\Gamma,X_\square},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,X_\square}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{3.1.575}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{3.1.576}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 3.1.96. Similar proposition holds for

$$\text{Quasicoherentsheaves, Perfectcomplex, IndBanach}_*. \tag{3.1.577}$$

3.1.4 Univariate Hodge Iwasawa Prestacks

Frobenius Quasicoherent Prestacks I

Definition 3.1.97. We now consider the pro-étale site of $\mathrm{Spa}\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$, denote that by $*$. To be more accurate we replace one component for Γ with the pro-étale site of $\mathrm{Spa}\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$. And we treat then all the functor to be prestacks for this site. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power[6]:

$$\widetilde{\Delta}_*, \widetilde{\nabla}_*, \widetilde{\Phi}_*, \widetilde{\Delta}^+_*, \widetilde{\nabla}^+_*, \widetilde{\Delta}^\dagger_*, \widetilde{\nabla}^\dagger_*, \widetilde{\Phi}^r_*, \widetilde{\Phi}^I_*,$$

$$\breve{\Delta}_*, \breve{\nabla}_*, \breve{\Phi}_*, \breve{\Delta}^+_*, \breve{\nabla}^+_*, \breve{\Delta}^\dagger_*, \breve{\nabla}^\dagger_*, \breve{\Phi}^r_*, \breve{\Phi}^I_*,$$

$$\Delta_*, \nabla_*, \Phi_*, \Delta^+_*, \nabla^+_*, \Delta^\dagger_*, \nabla^\dagger_*, \Phi^r_*, \Phi^I_*.$$

Taking the product we have:

$$\widetilde{\Phi}_{*,X}, \widetilde{\Phi}^r_{*,X}, \widetilde{\Phi}^I_{*,X},$$

$$\breve{\Phi}_{*,X}, \breve{\Phi}^r_{*,X}, \breve{\Phi}^I_{*,X},$$

$$\Phi_{*,X}, \Phi^r_{*,X}, \Phi^I_{*,X}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma := \mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty,1)$-categories of $(\infty,1)$-modules.

Definition 3.1.98. First we consider the Bambozzi-Kremnizer spectrum $\mathrm{Spec}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,X}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,X}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,X}, \tag{3.1.578}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,X}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,X}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,X}, \tag{3.1.579}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,X}, \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,X}, \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,X}. \tag{3.1.580}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.581}$$

$$\tag{3.1.582}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.583}$$

$$\tag{3.1.584}$$

[6]Here $|\Gamma| = 1$.

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,X}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.585}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,X}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 3.1.99. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_*, \widetilde{\nabla}_*, \widetilde{\Phi}_*, \widetilde{\Delta}^+_*, \widetilde{\nabla}^+_*, \widetilde{\Delta}^\dagger_*, \widetilde{\nabla}^\dagger_*, \widetilde{\Phi}^r_*, \widetilde{\Phi}^I_*,$$

$$\breve{\Delta}_*, \breve{\nabla}_*, \breve{\Phi}_*, \breve{\Delta}^+_*, \breve{\nabla}^+_*, \breve{\Delta}^\dagger_*, \breve{\nabla}^\dagger_*, \breve{\Phi}^r_*, \breve{\Phi}^I_*,$$

$$\Delta_*, \nabla_*, \Phi_*, \Delta^+_*, \nabla^+_*, \Delta^\dagger_*, \nabla^\dagger_*, \Phi^r_*, \Phi^I_*,$$

with X. Then we have the notations:

$$\widetilde{\Delta}_{*,X}, \widetilde{\nabla}_{*,X}, \widetilde{\Phi}_{*,X}, \widetilde{\Delta}^+_{*,X}, \widetilde{\nabla}^+_{*,X}, \widetilde{\Delta}^\dagger_{*,X}, \widetilde{\nabla}^\dagger_{*,X}, \widetilde{\Phi}^r_{*,X}, \widetilde{\Phi}^I_{*,X},$$

$$\breve{\Delta}_{*,X}, \breve{\nabla}_{*,X}, \breve{\Phi}_{*,X}, \breve{\Delta}^+_{*,X}, \breve{\nabla}^+_{*,X}, \breve{\Delta}^\dagger_{*,X}, \breve{\nabla}^\dagger_{*,X}, \breve{\Phi}^r_{*,X}, \breve{\Phi}^I_{*,X},$$

$$\Delta_{*,X}, \nabla_{*,X}, \Phi_{*,X}, \Delta^+_{*,X}, \nabla^+_{*,X}, \Delta^\dagger_{*,X}, \nabla^\dagger_{*,X}, \Phi^r_{*,X}, \Phi^I_{*,X}.$$

Definition 3.1.100. First we consider the Clausen-Scholze spectrum $\mathrm{Spec}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,X}, \tag{3.1.586}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,X}, \tag{3.1.587}$$

(3.1.588)

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,X}, \breve{\nabla}_{*,X}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,X}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,X}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,X}, \tag{3.1.589}$$
$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,X}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,X}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,X}, \breve{\Phi}^I_{*,X}, \tag{3.1.590}$$
$$\tag{3.1.591}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,X}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,X}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,X}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,X}, \mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,X}, \tag{3.1.592}$$
$$\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,X}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,X}, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,X}, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,X}. \tag{3.1.593}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.594}$$
$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.595}$$
$$\tag{3.1.596}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.597}$$
$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.598}$$
$$\tag{3.1.599}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.600}$$
$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,X}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.601}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,X}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{*,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 3.1.101. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\mathrm{Quasicoherentpresheaves}, \mathrm{IndBanach}_{*} \tag{3.1.602}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.603}$$

$$\tag{3.1.604}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.605}$$

$$\tag{3.1.606}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,X}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.607}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,X}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 3.1.102. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\mathrm{Quasicoherentsheaves}, \mathrm{Condensed}_{*} \tag{3.1.608}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.609}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.610}$$

$$\tag{3.1.611}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.612}$$
$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.613}$$
$$\tag{3.1.614}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.615}$$
$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,X}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.616}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\mathrm{homotopylimit}_r\, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,X}, \mathrm{homotopycolimit}_I\, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,X},$$

$$\mathrm{homotopylimit}_r\, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,X}, \mathrm{homotopycolimit}_I\, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,X},$$

$$\mathrm{homotopylimit}_r\, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,X}, \mathrm{homotopycolimit}_I\, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,X}.$$

$$\mathrm{homotopylimit}_r\, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{homotopycolimit}_I\, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\mathrm{homotopylimit}_r\, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{homotopycolimit}_I\, \breve{\Phi}^I_{*,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\mathrm{homotopylimit}_r\, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{homotopycolimit}_I\, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 3.1.103. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Condensed}_* \tag{3.1.617}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.618}$$
$$\tag{3.1.619}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.620}$$
$$\tag{3.1.621}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.622}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we

consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,X}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{*,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{3.1.623}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{3.1.624}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 3.1.104. Similar proposition holds for

$$\mathrm{Quasicoherentsheaves}, \mathrm{IndBanach}_*. \tag{3.1.625}$$

Definition 3.1.105. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Perfectcomplex}, \mathrm{IndBanach}_* \tag{3.1.626}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.627}$$

$$\tag{3.1.628}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.629}$$
$$\tag{3.1.630}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,X}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.631}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,X}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 3.1.106. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\mathrm{Quasicoherentsheaves, Perfectcomplex, Condensed}_{*} \tag{3.1.632}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.633}$$
$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.634}$$
$$\tag{3.1.635}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.636}$$
$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.637}$$
$$\tag{3.1.638}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.639}$$
$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,X}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.640}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,X}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{*,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 3.1.107. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Perfectcomplex}, \mathrm{Condensed}_* \tag{3.1.641}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.642}$$

(3.1.643)

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.644}$$

(3.1.645)

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.646}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,X}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{*,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{3.1.647}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{3.1.648}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 3.1.108. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{Perfectcomplex}, \text{IndBanach}_*. \tag{3.1.649}$$

Frobenius Quasicoherent Prestacks II: Deformation in Preadic Spaces

Definition 3.1.109. We now consider the pro-étale site of $\mathrm{Spa}\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$, denote that by $*$. To be more accurate we replace one component for Γ with the pro-étale site of $\mathrm{Spa}\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$. And we treat then all the functor to be prestacks for this site. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power[7]:

$$\widetilde{\Delta}_*, \widetilde{\nabla}_*, \widetilde{\Phi}_*, \widetilde{\Delta}^+_*, \widetilde{\nabla}^+_*, \widetilde{\Delta}^\dagger_*, \widetilde{\nabla}^\dagger_*, \widetilde{\Phi}^r_*, \widetilde{\Phi}^I_*,$$

$$\breve{\Delta}_*, \breve{\nabla}_*, \breve{\Phi}_*, \breve{\Delta}^+_*, \breve{\nabla}^+_*, \breve{\Delta}^\dagger_*, \breve{\nabla}^\dagger_*, \breve{\Phi}^r_*, \breve{\Phi}^I_*,$$

$$\Delta_*, \nabla_*, \Phi_*, \Delta^+_*, \nabla^+_*, \Delta^\dagger_*, \nabla^\dagger_*, \Phi^r_*, \Phi^I_*.$$

Taking the product we have:

$$\widetilde{\Phi}_{*,\circ}, \widetilde{\Phi}^r_{*,\circ}, \widetilde{\Phi}^I_{*,\circ},$$

$$\breve{\Phi}_{*,\circ}, \breve{\Phi}^r_{*,\circ}, \breve{\Phi}^I_{*,\circ},$$

$$\Phi_{*,\circ}, \Phi^r_{*,\circ}, \Phi^I_{*,\circ}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma := \mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty,1)$-categories of $(\infty,1)$-modules.

[7] Here $|\Gamma| = 1$.

Definition 3.1.110. First we consider the Bambozzi-Kremnizer spectrum $\mathrm{Spec}^{\mathrm{BK}}(*)$ attached to any of those in the above from [BK] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,\circ}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\circ}, \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\circ}, \tag{3.1.650}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,\circ}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\circ}, \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\circ}, \tag{3.1.651}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,\circ}, \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\circ}, \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\circ}. \tag{3.1.652}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.653}$$

$$\tag{3.1.654}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.655}$$

$$\tag{3.1.656}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.657}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\circ}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 3.1.111. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_*, \widetilde{\nabla}_*, \widetilde{\Phi}_*, \widetilde{\Delta}^+_*, \widetilde{\nabla}^+_*, \widetilde{\Delta}^\dagger_*, \widetilde{\nabla}^\dagger_*, \widetilde{\Phi}^r_*, \widetilde{\Phi}^I_*,$$

$$\breve{\Delta}_*, \breve{\nabla}_*, \breve{\Phi}_*, \breve{\Delta}^+_*, \breve{\nabla}^+_*, \breve{\Delta}^\dagger_*, \breve{\nabla}^\dagger_*, \breve{\Phi}^r_*, \breve{\Phi}^I_*,$$

$$\Delta_*, \nabla_*, \Phi_*, \Delta^+_*, \nabla^+_*, \Delta^\dagger_*, \nabla^\dagger_*, \Phi^r_*, \Phi^I_*,$$

with $\circ$. Then we have the notations:

$$\widetilde{\Delta}_{*,\circ}, \widetilde{\nabla}_{*,\circ}, \widetilde{\Phi}_{*,\circ}, \widetilde{\Delta}^+_{*,\circ}, \widetilde{\nabla}^+_{*,\circ}, \widetilde{\Delta}^\dagger_{*,\circ}, \widetilde{\nabla}^\dagger_{*,\circ}, \widetilde{\Phi}^r_{*,\circ}, \widetilde{\Phi}^I_{*,\circ},$$

$$\breve{\Delta}_{*,\circ}, \breve{\nabla}_{*,\circ}, \breve{\Phi}_{*,\circ}, \breve{\Delta}^+_{*,\circ}, \breve{\nabla}^+_{*,\circ}, \breve{\Delta}^\dagger_{*,\circ}, \breve{\nabla}^\dagger_{*,\circ}, \breve{\Phi}^r_{*,\circ}, \breve{\Phi}^I_{*,\circ},$$

$$\Delta_{*,\circ}, \nabla_{*,\circ}, \Phi_{*,\circ}, \Delta^+_{*,\circ}, \nabla^+_{*,\circ}, \Delta^\dagger_{*,\circ}, \nabla^\dagger_{*,\circ}, \Phi^r_{*,\circ}, \Phi^I_{*,\circ}.$$

Definition 3.1.112. First we consider the Clausen-Scholze spectrum $\mathrm{Spec}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,\circ}, \tag{3.1.658}$$
$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\circ}, \tag{3.1.659}$$
$$\tag{3.1.660}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\circ}, \breve{\nabla}_{*,\circ}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\circ}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,\circ}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,\circ}, \tag{3.1.661}$$
$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,\circ}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,\circ}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\circ}, \breve{\Phi}^I_{*,\circ}, \tag{3.1.662}$$
$$\tag{3.1.663}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\circ}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\circ}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\circ}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,\circ}, \mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,\circ}, \tag{3.1.664}$$
$$\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,\circ}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,\circ}, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\circ}, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\circ}. \tag{3.1.665}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.666}$$
$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.667}$$
$$\tag{3.1.668}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.669}$$
$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.670}$$
$$\tag{3.1.671}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.672}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.673}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r,\bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\circ},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\circ},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\circ},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\circ}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}},\underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 3.1.113. We then consider the corresponding quasipresheaves of the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\mathrm{Quasicoherentpresheaves},\mathrm{IndBanach}_{*} \tag{3.1.674}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.675}$$

$$\tag{3.1.676}$$

$$\mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.677}$$

$$\tag{3.1.678}$$

$$\mathrm{Spec}^{\mathrm{BK}}\Phi_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.679}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r,\bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\circ},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\circ},\underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\circ},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\circ}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\circ}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^r_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\widetilde{\Phi}^I_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^r_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\breve{\Phi}^I_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^r_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{BK}}\Phi^I_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 3.1.114. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Condensed}_* \tag{3.1.680}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.681}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.682}$$

$$\tag{3.1.683}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.684}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.685}$$

$$\tag{3.1.686}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.687}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.688}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\circ}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\circ},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\circ}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,\circ},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\circ}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\circ}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 3.1.115. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Condensed}_* \tag{3.1.689}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.690}$$

$$\tag{3.1.691}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.692}$$

$$\tag{3.1.693}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.694}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\circ}, \underset{I}{\text{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\circ},$$

$$\underset{r}{\text{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\circ}, \underset{I}{\text{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,\circ},$$

$$\underset{r}{\text{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\circ}, \underset{I}{\text{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\circ}.$$

$$\underset{r}{\text{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\, \breve{\Phi}^I_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\, M_r, \tag{3.1.695}$$

$$\underset{I}{\text{homotopylimit}}\, M_I, \tag{3.1.696}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 3.1.116. Similar proposition holds for

$$\text{Quasicoherentsheaves}, \text{IndBanach}_*. \tag{3.1.697}$$

Definition 3.1.117. We then consider the corresponding quasipresheaves of perfect complexes the corresponding ind-Banach or monomorphic ind-Banach modules from [BBK], [KKM]:

$$\text{Quasicoherentpresheaves}, \text{Perfectcomplex}, \text{IndBanach}_* \tag{3.1.698}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{BK}}\widetilde{\Phi}_{*,\circ}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.699}$$

$$\tag{3.1.700}$$

$$\text{Spec}^{\text{BK}}\breve{\Phi}_{*,\circ}/\text{Fro}^{\mathbb{Z}}, \tag{3.1.701}$$

$$\tag{3.1.702}$$

$$\text{Spec}^{\text{BK}}\Phi_{*,\circ}/\text{Fro}^{\mathbb{Z}}. \tag{3.1.703}$$

Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{*,\circ}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{*,\circ},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^r_{*,\circ}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^I_{*,\circ},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\Phi^r_{*,\circ}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\Phi^I_{*,\circ}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^r_{*,\circ}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\widetilde{\Phi}^I_{*,\circ}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^r_{*,\circ}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\breve{\Phi}^I_{*,\circ}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{BK}}\Phi^r_{*,\circ}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{BK}}\Phi^I_{*,\circ}/\text{Fro}^{\mathbb{Z}}.$$

Definition 3.1.118. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves}, \text{Perfectcomplex}, \text{Condensed}_* \tag{3.1.704}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^{+}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.705}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^{+}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^{\dagger}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^{\dagger}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.706}$$

$$\tag{3.1.707}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^{+}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.708}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^{+}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^{\dagger}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^{\dagger}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.709}$$

$$\tag{3.1.710}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^{+}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.711}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^{+}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^{\dagger}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^{\dagger}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.1.712}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{I}_{*,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{r}_{*,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{I}_{*,\circ}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{r}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^{I}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{r}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{I}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 3.1.119. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Perfectcomplex}, \mathrm{Condensed}_{*} \tag{3.1.713}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.714}$$

$$\tag{3.1.715}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.716}$$

$$\tag{3.1.717}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.1.718}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{*,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\circ}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{*,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 12.18]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{3.1.719}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{3.1.720}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Proposition 3.1.120. Similar proposition holds for

$$\mathrm{Quasicoherentsheaves}, \mathrm{Perfectcomplex}, \mathrm{IndBanach}_*. \tag{3.1.721}$$

3.2 Over Affinoid Analytic Spaces

This chapter follows closely [T1], [T2], [T3], [KPX], [KP], [KL1], [KL2], [BK], [BBBK], [BBM], [KKM], [CS1], [CS2], [CKZ], [PZ], [BCM], [LBV], [T3], [He], [PR], [SW], [FS], [RZ], [Sch2], where along one direction we will have the goal in mind to study the moduli stacks of Frobenius modules in some sense. [8] All the corresponding affinoid analytic spaces will be Clausen-Scholze spectra of analytic rings in [CS1] and [CS2], in the notation of X, $\circ$, $X_{\square}$.

Frobenius Quasicoherent Modules I

Definition 3.2.1. Let ψ be a toric tower over $\mathbb{Q}_p$ as in [KL2, Chapter 7] with base $\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power:

$$\widetilde{\Delta}_{\psi,\Gamma},\widetilde{\nabla}_{\psi,\Gamma},\widetilde{\Phi}_{\psi,\Gamma},\widetilde{\Delta}^+_{\psi,\Gamma},\widetilde{\nabla}^+_{\psi,\Gamma},\widetilde{\Delta}^\dagger_{\psi,\Gamma},\widetilde{\nabla}^\dagger_{\psi,\Gamma},\widetilde{\Phi}^r_{\psi,\Gamma},\widetilde{\Phi}^I_{\psi,\Gamma},$$

$$\breve{\Delta}_{\psi,\Gamma},\breve{\nabla}_{\psi,\Gamma},\breve{\Phi}_{\psi,\Gamma},\breve{\Delta}^+_{\psi,\Gamma},\breve{\nabla}^+_{\psi,\Gamma},\breve{\Delta}^\dagger_{\psi,\Gamma},\breve{\nabla}^\dagger_{\psi,\Gamma},\breve{\Phi}^r_{\psi,\Gamma},\breve{\Phi}^I_{\psi,\Gamma},$$

$$\Delta_{\psi,\Gamma},\nabla_{\psi,\Gamma},\Phi_{\psi,\Gamma},\Delta^+_{\psi,\Gamma},\nabla^+_{\psi,\Gamma},\Delta^\dagger_{\psi,\Gamma},\nabla^\dagger_{\psi,\Gamma},\Phi^r_{\psi,\Gamma},\Phi^I_{\psi,\Gamma}.$$

Taking the product we have:

$$\widetilde{\Phi}_{\psi,\Gamma,X},\widetilde{\Phi}^r_{\psi,\Gamma,X},\widetilde{\Phi}^I_{\psi,\Gamma,X},$$

$$\breve{\Phi}_{\psi,\Gamma,X},\breve{\Phi}^r_{\psi,\Gamma,X},\breve{\Phi}^I_{\psi,\Gamma,X},$$

$$\Phi_{\psi,\Gamma,X},\Phi^r_{\psi,\Gamma,X},\Phi^I_{\psi,\Gamma,X}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma := \mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty,1)$-categories of $(\infty,1)$-modules.

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 3.2.2. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_{\psi,\Gamma},\widetilde{\nabla}_{\psi,\Gamma},\widetilde{\Phi}_{\psi,\Gamma},\widetilde{\Delta}^+_{\psi,\Gamma},\widetilde{\nabla}^+_{\psi,\Gamma},\widetilde{\Delta}^\dagger_{\psi,\Gamma},\widetilde{\nabla}^\dagger_{\psi,\Gamma},\widetilde{\Phi}^r_{\psi,\Gamma},\widetilde{\Phi}^I_{\psi,\Gamma},$$

$$\breve{\Delta}_{\psi,\Gamma},\breve{\nabla}_{\psi,\Gamma},\breve{\Phi}_{\psi,\Gamma},\breve{\Delta}^+_{\psi,\Gamma},\breve{\nabla}^+_{\psi,\Gamma},\breve{\Delta}^\dagger_{\psi,\Gamma},\breve{\nabla}^\dagger_{\psi,\Gamma},\breve{\Phi}^r_{\psi,\Gamma},\breve{\Phi}^I_{\psi,\Gamma},$$

$$\Delta_{\psi,\Gamma},\nabla_{\psi,\Gamma},\Phi_{\psi,\Gamma},\Delta^+_{\psi,\Gamma},\nabla^+_{\psi,\Gamma},\Delta^\dagger_{\psi,\Gamma},\nabla^\dagger_{\psi,\Gamma},\Phi^r_{\psi,\Gamma},\Phi^I_{\psi,\Gamma},$$

with X. Then we have the notations:

$$\widetilde{\Delta}_{\psi,\Gamma,X},\widetilde{\nabla}_{\psi,\Gamma,X},\widetilde{\Phi}_{\psi,\Gamma,X},\widetilde{\Delta}^+_{\psi,\Gamma,X},\widetilde{\nabla}^+_{\psi,\Gamma,X},\widetilde{\Delta}^\dagger_{\psi,\Gamma,X},\widetilde{\nabla}^\dagger_{\psi,\Gamma,X},\widetilde{\Phi}^r_{\psi,\Gamma,X},\widetilde{\Phi}^I_{\psi,\Gamma,X},$$

[8] The consideration will be essentially after the work [EG-2], the work [HHS] and the work [EGH]. See [EGH, Conjecture 5.1.18, Section 5.2, Theorem, 5.2.4] for the detail of Emerton-Gee-Hellmann conjecture on the moduli stack of (φ,Γ)-modules.

$$\breve{\Delta}_{\psi,\Gamma,X}, \breve{\nabla}_{\psi,\Gamma,X}, \breve{\Phi}_{\psi,\Gamma,X}, \breve{\Delta}^+_{\psi,\Gamma,X}, \breve{\nabla}^+_{\psi,\Gamma,X}, \breve{\Delta}^\dagger_{\psi,\Gamma,X}, \breve{\nabla}^\dagger_{\psi,\Gamma,X}, \breve{\Phi}^r_{\psi,\Gamma,X}, \breve{\Phi}^I_{\psi,\Gamma,X},$$

$$\Delta_{\psi,\Gamma,X}, \nabla_{\psi,\Gamma,X}, \Phi_{\psi,\Gamma,X}, \Delta^+_{\psi,\Gamma,X}, \nabla^+_{\psi,\Gamma,X}, \Delta^\dagger_{\psi,\Gamma,X}, \nabla^\dagger_{\psi,\Gamma,X}, \Phi^r_{\psi,\Gamma,X}, \Phi^I_{\psi,\Gamma,X}.$$

Definition 3.2.3. First we consider the Clausen-Scholze spectrum $\mathrm{Spec}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,X}, \tag{3.2.1}$$
$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X}, \tag{3.2.2}$$
$$\tag{3.2.3}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,X}, \breve{\nabla}_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,X}, \tag{3.2.4}$$
$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X}, \breve{\Phi}^I_{\psi,\Gamma,X}, \tag{3.2.5}$$
$$\tag{3.2.6}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,X}, \tag{3.2.7}$$
$$\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X}, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X}. \tag{3.2.8}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.9}$$
$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.10}$$
$$\tag{3.2.11}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.12}$$
$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.13}$$
$$\tag{3.2.14}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.15}$$
$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.2.16}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,X},$$

$$\underset{r}{\text{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X}.$$

$$\underset{r}{\text{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\, \breve{\Phi}^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 3.2.4. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Condensed}_* \tag{3.2.17}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.18}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.19}$$

$$\tag{3.2.20}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.21}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.22}$$

$$\tag{3.2.23}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.24}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.2.25}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X},$$

$$\underset{r}{\text{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,X},$$

$$\underset{r}{\text{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X}.$$

$$\underset{r}{\text{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\, \breve{\Phi}^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 3.2.5. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Condensed}_* \tag{3.2.26}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.27}$$

$$\tag{3.2.28}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.29}$$

$$\tag{3.2.30}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.31}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X},$$

$$\underset{r}{\text{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,X},$$

$$\underset{r}{\text{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X}.$$

$$\underset{r}{\text{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\, \breve{\Phi}^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\, M_r, \tag{3.2.32}$$

$$\underset{I}{\text{homotopylimit}}\, M_I, \tag{3.2.33}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Definition 3.2.6. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Perfectcomplex, Condensed}_* \tag{3.2.34}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{CS}}\widetilde{\Delta}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Phi}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.2.35}$$

$$\text{Spec}^{\text{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.2.36}$$

$$\tag{3.2.37}$$

$$\text{Spec}^{\text{CS}}\breve{\Delta}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\breve{\Phi}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\breve{\Delta}^+_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.2.38}$$

$$\text{Spec}^{\text{CS}}\breve{\nabla}^+_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.2.39}$$

$$\tag{3.2.40}$$

$$\text{Spec}^{\text{CS}}\Delta_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\nabla_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\Phi_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\Delta^+_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.2.41}$$

$$\text{Spec}^{\text{CS}}\nabla^+_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\Delta^\dagger_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\nabla^\dagger_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}. \tag{3.2.42}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{\psi,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^I_{\psi,\Gamma,X},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{\psi,\Gamma,X}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{\psi,\Gamma,X}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}.$$

Proposition 3.2.7. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Perfectcomplex, Condensed}_* \tag{3.2.43}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{CS}}\widetilde{\Phi}_{\psi,\Gamma,X}/\text{Fro}^{\mathbb{Z}}, \tag{3.2.44}$$

$$\tag{3.2.45}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.46}$$

$$\tag{3.2.47}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.48}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,X},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{3.2.49}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{3.2.50}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Frobenius Quasicoherent Modules II: Moduli Stacks of Frobenius Modules

Definition 3.2.8. Let ψ be a toric tower over $\mathbb{Q}_p$ as in [KL2, Chapter 7] with base $\mathbb{Q}_p\left\langle X_1^{\pm 1},...,X_k^{\pm 1}\right\rangle$. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power:

$$\widetilde{\Delta}_{\psi,\Gamma}, \widetilde{\nabla}_{\psi,\Gamma}, \widetilde{\Phi}_{\psi,\Gamma}, \widetilde{\Delta}^+_{\psi,\Gamma}, \widetilde{\nabla}^+_{\psi,\Gamma}, \widetilde{\Delta}^\dagger_{\psi,\Gamma}, \widetilde{\nabla}^\dagger_{\psi,\Gamma}, \widetilde{\Phi}^r_{\psi,\Gamma}, \widetilde{\Phi}^I_{\psi,\Gamma},$$

$$\breve{\Delta}_{\psi,\Gamma}, \breve{\nabla}_{\psi,\Gamma}, \breve{\Phi}_{\psi,\Gamma}, \breve{\Delta}^+_{\psi,\Gamma}, \breve{\nabla}^+_{\psi,\Gamma}, \breve{\Delta}^\dagger_{\psi,\Gamma}, \breve{\nabla}^\dagger_{\psi,\Gamma}, \breve{\Phi}^r_{\psi,\Gamma}, \breve{\Phi}^I_{\psi,\Gamma},$$

$$\Delta_{\psi,\Gamma}, \nabla_{\psi,\Gamma}, \Phi_{\psi,\Gamma}, \Delta^+_{\psi,\Gamma}, \nabla^+_{\psi,\Gamma}, \Delta^\dagger_{\psi,\Gamma}, \nabla^\dagger_{\psi,\Gamma}, \Phi^r_{\psi,\Gamma}, \Phi^I_{\psi,\Gamma}.$$

Taking the product we have:

$$\widetilde{\Phi}_{\psi,\Gamma,\circ}, \widetilde{\Phi}^r_{\psi,\Gamma,\circ}, \widetilde{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\breve{\Phi}_{\psi,\Gamma,\circ}, \breve{\Phi}^r_{\psi,\Gamma,\circ}, \breve{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\Phi_{\psi,\Gamma,\circ}, \Phi^r_{\psi,\Gamma,\circ}, \Phi^I_{\psi,\Gamma,\circ}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma := \mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty,1)$-categories of $(\infty,1)$-modules.

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 3.2.9. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_{\psi,\Gamma}, \widetilde{\nabla}_{\psi,\Gamma}, \widetilde{\Phi}_{\psi,\Gamma}, \widetilde{\Delta}^+_{\psi,\Gamma}, \widetilde{\nabla}^+_{\psi,\Gamma}, \widetilde{\Delta}^\dagger_{\psi,\Gamma}, \widetilde{\nabla}^\dagger_{\psi,\Gamma}, \widetilde{\Phi}^r_{\psi,\Gamma}, \widetilde{\Phi}^I_{\psi,\Gamma},$$

$$\breve{\Delta}_{\psi,\Gamma}, \breve{\nabla}_{\psi,\Gamma}, \breve{\Phi}_{\psi,\Gamma}, \breve{\Delta}^+_{\psi,\Gamma}, \breve{\nabla}^+_{\psi,\Gamma}, \breve{\Delta}^\dagger_{\psi,\Gamma}, \breve{\nabla}^\dagger_{\psi,\Gamma}, \breve{\Phi}^r_{\psi,\Gamma}, \breve{\Phi}^I_{\psi,\Gamma},$$

$$\Delta_{\psi,\Gamma}, \nabla_{\psi,\Gamma}, \Phi_{\psi,\Gamma}, \Delta^+_{\psi,\Gamma}, \nabla^+_{\psi,\Gamma}, \Delta^\dagger_{\psi,\Gamma}, \nabla^\dagger_{\psi,\Gamma}, \Phi^r_{\psi,\Gamma}, \Phi^I_{\psi,\Gamma},$$

with $\circ$. Then we have the notations:

$$\widetilde{\Delta}_{\psi,\Gamma,\circ}, \widetilde{\nabla}_{\psi,\Gamma,\circ}, \widetilde{\Phi}_{\psi,\Gamma,\circ}, \widetilde{\Delta}^+_{\psi,\Gamma,\circ}, \widetilde{\nabla}^+_{\psi,\Gamma,\circ}, \widetilde{\Delta}^\dagger_{\psi,\Gamma,\circ}, \widetilde{\nabla}^\dagger_{\psi,\Gamma,\circ}, \widetilde{\Phi}^r_{\psi,\Gamma,\circ}, \widetilde{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\breve{\Delta}_{\psi,\Gamma,\circ}, \breve{\nabla}_{\psi,\Gamma,\circ}, \breve{\Phi}_{\psi,\Gamma,\circ}, \breve{\Delta}^+_{\psi,\Gamma,\circ}, \breve{\nabla}^+_{\psi,\Gamma,\circ}, \breve{\Delta}^\dagger_{\psi,\Gamma,\circ}, \breve{\nabla}^\dagger_{\psi,\Gamma,\circ}, \breve{\Phi}^r_{\psi,\Gamma,\circ}, \breve{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\Delta_{\psi,\Gamma,\circ}, \nabla_{\psi,\Gamma,\circ}, \Phi_{\psi,\Gamma,\circ}, \Delta^+_{\psi,\Gamma,\circ}, \nabla^+_{\psi,\Gamma,\circ}, \Delta^\dagger_{\psi,\Gamma,\circ}, \nabla^\dagger_{\psi,\Gamma,\circ}, \Phi^r_{\psi,\Gamma,\circ}, \Phi^I_{\psi,\Gamma,\circ}.$$

Definition 3.2.10. First we consider the Clausen-Scholze spectrum $\mathrm{Spec}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,\circ}, \tag{3.2.51}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\circ}, \tag{3.2.52}$$

$$\tag{3.2.53}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,\circ}, \breve{\nabla}_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,\circ}, \tag{3.2.54}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\circ}, \breve{\Phi}^I_{\psi,\Gamma,\circ}, \tag{3.2.55}$$

$$\tag{3.2.56}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,\circ}, \tag{3.2.57}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\circ}, \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\circ}. \tag{3.2.58}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.59}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.60}$$

$$\tag{3.2.61}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.62}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.63}$$

$$\tag{3.2.64}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.65}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.2.66}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\circ}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 3.2.11. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Condensed}_{*} \tag{3.2.67}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^{+}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.68}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^{+}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^{\dagger}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^{\dagger}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.69}$$

$$\tag{3.2.70}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^{+}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.71}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^{+}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^{\dagger}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^{\dagger}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.72}$$

$$\tag{3.2.73}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^{+}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.74}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^{+}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^{\dagger}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^{\dagger}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.2.75}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{\psi,\Gamma,\circ}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{\psi,\Gamma,\circ},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{r}_{\psi,\Gamma,\circ}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{I}_{\psi,\Gamma,\circ},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{r}_{\psi,\Gamma,\circ}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{I}_{\psi,\Gamma,\circ}.$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{r}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^{I}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{r}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{I}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 3.2.12. There is a well-defined functor from the ∞-category

$$\text{Quasicoherentpresheaves, Condensed}_{*} \tag{3.2.76}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.77}$$

$$\tag{3.2.78}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.79}$$

$$\tag{3.2.80}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.81}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\circ}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{3.2.82}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{3.2.83}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Definition 3.2.13. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\mathrm{Quasicoherentsheaves, Perfectcomplex, Condensed}_* \tag{3.2.84}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^{+}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.85}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^{+}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^{\dagger}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^{\dagger}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.86}$$

$$\tag{3.2.87}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^{+}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.88}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^{+}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^{\dagger}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^{\dagger}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.89}$$

$$\tag{3.2.90}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^{+}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.91}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^{+}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^{\dagger}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^{\dagger}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.2.92}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{\psi,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{r}_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{I}_{\psi,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{r}_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{I}_{\psi,\Gamma,\circ}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{r}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^{I}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^{r}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^{I}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{r}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^{I}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 3.2.14. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Perfectcomplex}, \mathrm{Condensed}_{*} \tag{3.2.93}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.94}$$

$$\tag{3.2.95}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.96}$$

$$\tag{3.2.97}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.98}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,\circ},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\circ}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\circ}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,\circ}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{3.2.99}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{3.2.100}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Frobenius Quasicoherent Modules III: Deformation in $(\infty, 1)$-Analytic Spaces

Definition 3.2.15. Let ψ be a toric tower over $\mathbb{Q}_p$ as in [KL2, Chapter 7] with base $\mathbb{Q}_p\left\langle X_1^{\pm 1}, ..., X_k^{\pm 1}\right\rangle$. Then from [KL1] and [KL2, Definition 5.2.1] we have the following class of Kedlaya-Liu rings (with the following replacement: Δ stands for A, ∇ stands for B, while Φ stands for C) by taking product in the sense of self Γ-th power:

$$\widetilde{\Delta}_{\psi,\Gamma}, \widetilde{\nabla}_{\psi,\Gamma}, \widetilde{\Phi}_{\psi,\Gamma}, \widetilde{\Delta}^+_{\psi,\Gamma}, \widetilde{\nabla}^+_{\psi,\Gamma}, \widetilde{\Delta}^\dagger_{\psi,\Gamma}, \widetilde{\nabla}^\dagger_{\psi,\Gamma}, \widetilde{\Phi}^r_{\psi,\Gamma}, \widetilde{\Phi}^I_{\psi,\Gamma},$$

$$\breve{\Delta}_{\psi,\Gamma}, \breve{\nabla}_{\psi,\Gamma}, \breve{\Phi}_{\psi,\Gamma}, \breve{\Delta}^+_{\psi,\Gamma}, \breve{\nabla}^+_{\psi,\Gamma}, \breve{\Delta}^\dagger_{\psi,\Gamma}, \breve{\nabla}^\dagger_{\psi,\Gamma}, \breve{\Phi}^r_{\psi,\Gamma}, \breve{\Phi}^I_{\psi,\Gamma},$$

$$\Delta_{\psi,\Gamma},\nabla_{\psi,\Gamma},\Phi_{\psi,\Gamma},\Delta^+_{\psi,\Gamma},\nabla^+_{\psi,\Gamma},\Delta^\dagger_{\psi,\Gamma},\nabla^\dagger_{\psi,\Gamma},\Phi^r_{\psi,\Gamma},\Phi^I_{\psi,\Gamma}.$$

Taking the product we have:

$$\widetilde{\Phi}_{\psi,\Gamma,X_\square},\widetilde{\Phi}^r_{\psi,\Gamma,X_\square},\widetilde{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\breve{\Phi}_{\psi,\Gamma,X_\square},\breve{\Phi}^r_{\psi,\Gamma,X_\square},\breve{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\Phi_{\psi,\Gamma,X_\square},\Phi^r_{\psi,\Gamma,X_\square},\Phi^I_{\psi,\Gamma,X_\square}.$$

They carry multi Frobenius action φ_Γ and multi $\mathrm{Lie}_\Gamma := \mathbb{Z}_p^{\times\Gamma}$ action. In our current situation after [CKZ] and [PZ] we consider the following $(\infty,1)$-categories of $(\infty,1)$-modules.

Meanwhile we have the corresponding Clausen-Scholze analytic stacks from [CS2], therefore applying their construction we have:

Definition 3.2.16. Here we define the following products by using the solidified tensor product from [CS1] and [CS2]. Then we take solidified tensor product $\overset{\blacksquare}{\otimes}$ of any of the following

$$\widetilde{\Delta}_{\psi,\Gamma},\widetilde{\nabla}_{\psi,\Gamma},\widetilde{\Phi}_{\psi,\Gamma},\widetilde{\Delta}^+_{\psi,\Gamma},\widetilde{\nabla}^+_{\psi,\Gamma},\widetilde{\Delta}^\dagger_{\psi,\Gamma},\widetilde{\nabla}^\dagger_{\psi,\Gamma},\widetilde{\Phi}^r_{\psi,\Gamma},\widetilde{\Phi}^I_{\psi,\Gamma},$$

$$\breve{\Delta}_{\psi,\Gamma},\breve{\nabla}_{\psi,\Gamma},\breve{\Phi}_{\psi,\Gamma},\breve{\Delta}^+_{\psi,\Gamma},\breve{\nabla}^+_{\psi,\Gamma},\breve{\Delta}^\dagger_{\psi,\Gamma},\breve{\nabla}^\dagger_{\psi,\Gamma},\breve{\Phi}^r_{\psi,\Gamma},\breve{\Phi}^I_{\psi,\Gamma},$$

$$\Delta_{\psi,\Gamma},\nabla_{\psi,\Gamma},\Phi_{\psi,\Gamma},\Delta^+_{\psi,\Gamma},\nabla^+_{\psi,\Gamma},\Delta^\dagger_{\psi,\Gamma},\nabla^\dagger_{\psi,\Gamma},\Phi^r_{\psi,\Gamma},\Phi^I_{\psi,\Gamma},$$

with $X_\square$. Then we have the notations:

$$\widetilde{\Delta}_{\psi,\Gamma,X_\square},\widetilde{\nabla}_{\psi,\Gamma,X_\square},\widetilde{\Phi}_{\psi,\Gamma,X_\square},\widetilde{\Delta}^+_{\psi,\Gamma,X_\square},\widetilde{\nabla}^+_{\psi,\Gamma,X_\square},\widetilde{\Delta}^\dagger_{\psi,\Gamma,X_\square},\widetilde{\nabla}^\dagger_{\psi,\Gamma,X_\square},\widetilde{\Phi}^r_{\psi,\Gamma,X_\square},\widetilde{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\breve{\Delta}_{\psi,\Gamma,X_\square},\breve{\nabla}_{\psi,\Gamma,X_\square},\breve{\Phi}_{\psi,\Gamma,X_\square},\breve{\Delta}^+_{\psi,\Gamma,X_\square},\breve{\nabla}^+_{\psi,\Gamma,X_\square},\breve{\Delta}^\dagger_{\psi,\Gamma,X_\square},\breve{\nabla}^\dagger_{\psi,\Gamma,X_\square},\breve{\Phi}^r_{\psi,\Gamma,X_\square},\breve{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\Delta_{\psi,\Gamma,X_\square},\nabla_{\psi,\Gamma,X_\square},\Phi_{\psi,\Gamma,X_\square},\Delta^+_{\psi,\Gamma,X_\square},\nabla^+_{\psi,\Gamma,X_\square},\Delta^\dagger_{\psi,\Gamma,X_\square},\nabla^\dagger_{\psi,\Gamma,X_\square},\Phi^r_{\psi,\Gamma,X_\square},\Phi^I_{\psi,\Gamma,X_\square}.$$

Definition 3.2.17. First we consider the Clausen-Scholze spectrum $\mathrm{Spec}^{\mathrm{CS}}(*)$ attached to any of those in the above from [CS2] by taking derived rational localization:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,X_\square},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,X_\square},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,X_\square},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,X_\square},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,X_\square}, \tag{3.2.101}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,X_\square},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,X_\square},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square}, \tag{3.2.102}$$

$$\tag{3.2.103}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,X_\square},\breve{\nabla}_{\psi,\Gamma,X_\square},\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,X_\square},\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,X_\square},\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,X_\square}, \tag{3.2.104}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,X_\square},\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,X_\square},\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X_\square},\breve{\Phi}^I_{\psi,\Gamma,X_\square}, \tag{3.2.105}$$

$$\tag{3.2.106}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,X_\square},\mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,X_\square},\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,X_\square},\mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,X_\square},\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,X_\square}, \tag{3.2.107}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,X_\square},\mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,X_\square},\mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X_\square},\mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X_\square}. \tag{3.2.108}$$

Then we take the corresponding quotients by using the corresponding Frobenius operators:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.109}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.110}$$

$$\tag{3.2.111}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\breve{\nabla}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.112}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.113}$$

$$\tag{3.2.114}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.115}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.2.116}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r,\bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\,\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square},\underset{I}{\mathrm{homotopycolimit}}\,\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\,\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X_\square},\underset{I}{\mathrm{homotopycolimit}}\,\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\,\mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X_\square},\underset{I}{\mathrm{homotopycolimit}}\,\mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X_\square}.$$

$$\underset{r}{\mathrm{homotopylimit}}\,\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\underset{I}{\mathrm{homotopycolimit}}\,\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\,\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\underset{I}{\mathrm{homotopycolimit}}\,\breve{\Phi}^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\,\mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\underset{I}{\mathrm{homotopycolimit}}\,\mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

Definition 3.2.18. We then consider the corresponding quasisheaves of the corresponding condensed solid topological modules from [CS2]:

$$\mathrm{Quasicoherentsheaves},\mathrm{Condensed}_* \tag{3.2.117}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.118}$$

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},\mathrm{Spec}^{\mathrm{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.119}$$

$$\tag{3.2.120}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z}, \breve{\nabla}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z}, \tag{3.2.121}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z}, \tag{3.2.122}$$

$$\tag{3.2.123}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z}, \tag{3.2.124}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z}. \tag{3.2.125}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X_\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X_\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X_\square}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z}.$$

Proposition 3.2.19. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Condensed}_* \tag{3.2.126}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z}, \tag{3.2.127}$$

$$\tag{3.2.128}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z}, \tag{3.2.129}$$

$$\tag{3.2.130}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,X_\square}/\mathrm{Fro}^\mathbb{Z}, \tag{3.2.131}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we

consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{\psi,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{\psi,\Gamma,X_\square}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{\psi,\Gamma,X_\square}.$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\breve{\Phi}^r_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\text{homotopylimit}}\ \text{Spec}^{\text{CS}}\Phi^r_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \underset{I}{\text{homotopycolimit}}\ \text{Spec}^{\text{CS}}\Phi^I_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. In this situation for modules parametrized by the intervals we have the equivalence of ∞-categories by using [CS2, Proposition 13.8]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\text{homotopycolimit}}\ M_r, \tag{3.2.132}$$

$$\underset{I}{\text{homotopylimit}}\ M_I, \tag{3.2.133}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Definition 3.2.20. We then consider the corresponding quasisheaves of perfect complexes of the corresponding condensed solid topological modules from [CS2]:

$$\text{Quasicoherentsheaves, Perfectcomplex, Condensed}_* \tag{3.2.134}$$

where $*$ is one of the following spaces:

$$\text{Spec}^{\text{CS}}\widetilde{\Delta}_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Phi}_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^+_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \tag{3.2.135}$$

$$\text{Spec}^{\text{CS}}\widetilde{\nabla}^+_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\Delta}^\dagger_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\widetilde{\nabla}^\dagger_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \tag{3.2.136}$$

$$\tag{3.2.137}$$

$$\text{Spec}^{\text{CS}}\breve{\Delta}_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \breve{\nabla}_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\breve{\Phi}_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\breve{\Delta}^+_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \tag{3.2.138}$$

$$\text{Spec}^{\text{CS}}\breve{\nabla}^+_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\breve{\Delta}^\dagger_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \text{Spec}^{\text{CS}}\breve{\nabla}^\dagger_{\psi,\Gamma,X_\square}/\text{Fro}^{\mathbb{Z}}, \tag{3.2.139}$$

$$\tag{3.2.140}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Delta_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.141}$$

$$\mathrm{Spec}^{\mathrm{CS}}\nabla^+_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\Delta^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \mathrm{Spec}^{\mathrm{CS}}\nabla^\dagger_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}. \tag{3.2.142}$$

Here for those space with notations related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X_\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X_\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X_\square}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

Proposition 3.2.21. There is a well-defined functor from the ∞-category

$$\mathrm{Quasicoherentpresheaves}, \mathrm{Perfectcomplex}, \mathrm{Condensed}_* \tag{3.2.143}$$

where $*$ is one of the following spaces:

$$\mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.144}$$

$$\tag{3.2.145}$$

$$\mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.146}$$

$$\tag{3.2.147}$$

$$\mathrm{Spec}^{\mathrm{CS}}\Phi_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \tag{3.2.148}$$

to the ∞-category of Fro-equivariant quasicoherent presheaves over similar spaces above correspondingly without the Fro-quotients, and to the ∞-category of Fro-equivariant quasicoherent modules over global sections of the structure ∞-sheaves of the similar spaces above correspondingly without the Fro-quotients. Here for those space without notation related to the radius and the corresponding interval we consider the total unions $\bigcap_r, \bigcup_I$ in order to achieve the whole spaces to achieve the analogues of the corresponding FF curves from [KL1], [KL2], [FF] for

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X_\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^I_{\psi,\Gamma,X_\square},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X_\square}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X_\square}.$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\widetilde{\Phi}^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\breve{\Phi}^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \breve{\Phi}^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}},$$

$$\underset{r}{\mathrm{homotopylimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^r_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}, \underset{I}{\mathrm{homotopycolimit}}\ \mathrm{Spec}^{\mathrm{CS}}\Phi^I_{\psi,\Gamma,X_\square}/\mathrm{Fro}^{\mathbb{Z}}.$$

In this situation we will have the target category being family parametrized by r or I in compatible glueing sense as in [KL2, Definition 5.4.10]. Here the corresponding quasicoherent Frobenius modules are defined to be the corresponding homotopy colimits and limits of Frobenius modules:

$$\underset{r}{\mathrm{homotopycolimit}}\ M_r, \tag{3.2.149}$$

$$\underset{I}{\mathrm{homotopylimit}}\ M_I, \tag{3.2.150}$$

where each M_r is a Frobenius-equivariant module over the period ring with respect to some radius r while each M_I is a Frobenius-equivariant module over the period ring with respect to some interval I.

Acknowledgements

From the discussion with Professor Kedlaya, we understand the importance of the application of [BK], [BBBK], [BBK], [KKM], [BBM], [CS1], [CS2] to many problems in our setting. We thank Professor Kedlaya for the related discussion on these topics. We also have the chance to learn much from the work [EGH] through the discussion with the authors.

Bibliography

[KL1] Kedlaya, Kiran S., and Ruochuan Liu. "Relative p-adic Hodge theory: foundations." Astérisque 371, 2015.

[KL2] Kedlaya, Kiran S., and Ruochuan Liu. "Relative p-adic Hodge theory, II: Imperfect period rings." arXiv preprint arXiv:1602.06899 (2016).

[BBBK] Bambozzi, Federico, Oren Ben-Bassat, and Kobi Kremnizer. "Analytic geometry over $\mathbb{F}_1$ and the Fargues-Fontaine curve." Advances in Mathematics 356 (2019): 106815.

[BK] Bambozzi, Federico, and Kobi Kremnizer. "On the Sheafyness Property of Spectra of Banach Rings." arXiv preprint arXiv:2009.13926 (2020).

[BBK] Ben-Bassat, Oren, and Kobi Kremnizer. "Fréchet Modules and Descent." arXiv preprint arXiv:2002.11608 (2020).

[CS2] Clausen, Dustin, and Peter Scholze. Lectures on Analytic Geometry. Https://www.math.uni-bonn.de/people/scholze/Notes.html.

[CS1] Clausen, Dustin, and Peter Scholze. Lectures on Condensed Mathematics. Https://www.math.uni-bonn.de/people/scholze/Notes.html.

[BBM] Ben-Bassat, Oren, and Devarshi Mukherjee. 2022. "Analytification, Localization and Homotopy Epimorphisms." Bulletin Des Sciences Mathématiques 176: 103129. https://doi.org/10.1016/j.bulsci.2022.103129.

[KKM] Kelly, Jack, Kobi Kremnizer and Devarshi Mukherjee. "Analytic Hochschild-Kostant-Rosenberg Theorem." arxiv preprint arXiv:2111.03502.

[CKZ] Carter, Annie, Kiran S. Kedlaya and Gergely Zábrádi. "Drinfeld's Lemma for Perfectoid Spaces and Overconvergence of Multivariate (φ, Γ)-Modules." Doc. Math. 26 (2021), 1329-1393.

[Sch] Scholze, Peter. "p-adic Hodge theory for rigid-analytic varieties." Forum of Mathematics. Pi 1. 2013. https://doi.org/10.1017/fmp.2013.1.

[PZ] Pal, A., and Zábrádi, G. (2021). "Cohomology and Overconvergence for Representations of Powers of Galois Groups." Journal of the Institute of Mathematics of Jussieu, 20 (2), 361-421.

[T1] Tong, Xin. "Analytic geometry and Hodge-Frobenius Structure." arXiv:2011.08358.

[T2] Tong, Xin. "∞-Categorical Approaches to Hodge-Iwasawa Theory I: Introduction and Extensions." arXiv:2201.01979.

[T3] Tong, Xin. "Hodge-Iwasawa Theory II." arXiv:2010.02093.

[He] Hellmann, Eugen. "On arithmetic families of filtered φ-modules and crystalline representations." Journal of the Institute of Mathematics of Jussieu 12.04(2013):677-726.

[PR] Pappas, G., and M. Rapoport. "Phi-modules and coefficient spaces." Moscow Mathematical Journal 9.3(2008):625-663.

[SW] Scholze, Peter and Weinstein Jared. "Berkeley Lectures on p-adic Geometry." Princeton University Press, 2020.

[FS] Fargues, L., and P. Scholze. "Geometrization of the Local Langlands Correspondence." (2021). arXiv:2102.13459.

[RZ] Rapoport, M., and T. Zink. "Period Spaces for p-divisible Groups (AM-141), Volume 141." Princeton University Press. 2016.

[Sch2] Scholze, Peter. "Etale Cohomology of Diamonds." arXiv:1709.07343.

[BF1] Burns, D., and M. Flach. "Tamagawa Numbers for Motives with (Non-Commutative) Coefficients." Documenta mathematica Journal der Deutschen Mathematiker-Vereinigung 6.3(2003):475-512.

[BF2] Burns, D, and M Flach. 2003. "Tamagawa Numbers for Motives with (Noncommutative) Coefficients, II." American Journal of Mathematics 125 (3): 475-512. https://doi.org/10.1353/ajm.2003.0014.

[FK] Fukaya, Takako and Kato, Kazuya. "A Formulation of Conjectures on p-adic Zeta Functions in Noncommutative Iwasawa Theory." Proceedings of the St. Petersburg Mathematical Society. Volume XII.

[Na1] Nakamura, Kentaro . "Iwasawa Theory of de Rham (φ,Γ)-Modules over the Robba Rings." Journal of the Institute of Mathematics of Jussieu 13.1(2014):65-118.

[Na2] Nakamura, Kentaro. (2017). "A Generalization of Kato's Local ε-Conjecture for (φ,Γ)-Modules over the Robba Ring." Algebra and Number Theory 11. 2017, 319-404.

[Wit] Witte, M. "Noncommutative Iwasawa Main Conjectures for Varieties over Finite Fields." PhD dissertation, 2008.

[KP] Kedlaya, K. S., and J. Pottharst. "On Categories of (φ,Γ)-Modules." Proceedings of Symposia in Pure Mathematics 97, AMS, Providence, 2018, 281-304.

[FF] Fargues, Laurent, Jean Marc Fontaine. "Courbes et Fibrés Vectoriels en Théorie de Hodge p-adique." Astérisque 406 (2018): 1-382.

[KPX] Kedlaya, K. S., J. Pottharst, and L. Xiao. "Cohomology of Arithmetic Families of (φ,Γ)-Modules." Journal of the American Mathematical Society 27.4 (2012):1043-1115.

[T4] Tong, Xin. "Period Rings with Big Coefficients and Applications I." arXiv:2012.07338.

[T5] Tong, Xin. "Period Rings with Big Coefficients and Application II." arXiv:2101.03748.

[T6] Tong, Xin. "Period Rings with Big Coefficients and Applications III." arXiv:2102.10692.

[Hu1] Huber, R. "A Generalization of Formal Schemes and Rigid Analytic Varieties." Mathematische Zeitschrift 217.1(1994):513-551.

[Hu2] Huber, R. "Étale Cohomology of Rigid Analytic Spaces and Adic Spaces." 1996, Aspects of Mathematics, Vol 30.

[LBV] Le Bras, Arthur-César and Alberto Vezzani. "On de Rham-Fargues-Fontaine Cohomology." arXiv:2105.13028.

[BCM] Brinon, Olivier, Bruno Chiarellotto and Nicola Mazzari. "Multivariable de Rham representations, Sen theory and p-adic differential equations." (2021). arXiv:2111.11563.

[KL1-2] Kedlaya, Kiran S., and Ruochuan Liu. "Relative p-adic Hodge theory: foundations." Astérisque 371, 2015.

[KL2-2] Kedlaya, Kiran S., and Ruochuan Liu. "Relative p-adic Hodge theory, II: Imperfect period rings." arXiv preprint arXiv:1602.06899 (2016).

[FF-2] Fargues, Laurent, Jean Marc Fontaine. "Courbes et Fibrés Vectoriels en Théorie de Hodge p-adique." Astérisque 406 (2018): 1-382.

[T1-2] Tong, Xin. "Hodge-Iwasawa Theory I." arXiv:2006.03692.

[T2-2] Tong, Xin. "Hodge-Iwasawa Theory II." arXiv:2010.02093.

[BF1-2] Burns, D., and M. Flach. "Tamagawa Numbers for Motives with (Non-Commutative) Coefficients." Documenta mathematica Journal der Deutschen Mathematiker-Vereinigung 6.3(2003):475-512.

[BF2-2] Burns, D, and M Flach. 2003. "Tamagawa Numbers for Motives with (Non-commutative) Coefficients, II." American Journal of Mathematics 125 (3): 475-512. https://doi.org/10.1353/ajm.2003.0014.

[FK-2] Fukaya, Takako and Kato, Kazuya. "A Formulation of Conjectures on p-adic Zeta Functions in Noncommutative Iwasawa Theory." Proceedings of the St. Petersburg Mathematical Society Volume XII.

[KPX-2] Kedlaya, K. S., J. Pottharst, and L. Xiao. "Cohomology of Arithmetic Families of (φ,Γ)-Modules." Journal of the American Mathematical Society 27.4 (2012):1043-1115.

[KP-2] Kedlaya, K. S., and J. Pottharst. "On Categories of (φ,Γ)-Modules." Proceedings of Symposia in Pure Mathematics 97, AMS, Providence, 2018, 281-304.

[He1-2] Hellmann, Eugen. "On arithmetic families of filtered φ-modules and crystalline representations." Journal of the Institute of Mathematics of Jussieu 12.04(2013):677-726.

[HH-2] Hartl, Urs and Eugen Hellmann. "The universal family of semistable p-adic Galois representations." Algebra and Number Theory 14 (2020): 1055-1121.

[PR-2] Pappas, G., and M. Rapoport. "Phi-modules and coefficient spaces." Moscow Mathematical Journal 9.3(2008):625-663.

[RZ-2] Rapoport, M., and T. Zink. "Period Spaces for p-divisible Groups (AM-141), Volume 141." Annals of Mathematics Studies, Princeton University Press. 1996.

[SW-2] Scholze, Peter and Weinstein Jared. "Berkeley Lectures on p-adic Geometry." Princeton University Press, 2020.

[FS-2] Fargues, L., and P. Scholze. "Geometrization of the Local Langlands Correspondence." (2021). arXiv:2102.13459.

[Sch-2] Scholze, Peter. "p-adic Hodge theory for rigid-analytic varieties." Forum of Mathematics. Pi 1. 2013. https://doi.org/10.1017/fmp.2013.1.

[CS1-2] Clausen, Dustin, and Peter Scholze. Lectures on Condensed Mathematics. Https://www.math.uni-bonn.de/people/scholze/Notes.html.

[CS2-2] Clausen, Dustin, and Peter Scholze. Lectures on Analytic Geometry. Https://www.math.uni-bonn.de/people/scholze/Notes.html.

[Ta-2] Tate, John. "Rigid analytic spaces." Inventiones mathematicae 12 (1971): 257-289.

[Ked-2] Kedlaya, Kiran. "Relative p-adic Hodge Theory and Rapoport-Zink Period Domains." Proceedings of the International Congress of Mathematicians 2010, volume 2, 2010. 258-279

[Ked1-2] Kedlaya, Kiran. (2001). "A p-adic local monodromy theorem." Annals of Mathematics 160, 2004, 93-184.

[Ked2-2] Kedlaya, Kiran S. 2005. "local monodromy of p-adic differential equations: an overview." International Journal of Number Theory 1 (1): 109-154. https://doi.org/10.1142/S179304210500008X.

[Laff-2] Lafforgue, Vincent. 2018. "Chtoucas Pour Les Groupes Réductifs et Paramétrisation de Langlands Globale." Journal of the American Mathematical Society. Vol. 31. American Mathematical Society. https://doi.org/10.1090/jams/897.

[HV-2] Hartl, Urs, and Eva Viehmann. 2011. "The Newton Stratification on Deformations of Local G-Shtukas." Journal Für Die Reine Und Angewandte Mathematik 2011 (656): 87-129. https://doi.org/10.1515/crelle.2011.044.

[GL-2] Genestier, Alain , and V. Lafforgue. "Chtoucas restreints pour les groupes réductifs et paramétrisation de Langlands locale." arXiv:1709.00978.

[Har1-2] Hartl, Urs. 2005. "Uniformizing the Stacks of Abelian Sheaves." Number Fields and Function Fields-Two Parallel Worlds, 167-222. https://doi.org/10.1007/0-8176-4447-4_9.

[Har2-2] Hartl, Urs. 2011. "Period Spaces for Hodge Structures in Equal Characteristic." Annals of Mathematics 173 (3): 1241-1358. https://doi.org/10.4007/annals.2011.173.3.2.

[EG-2] Emerton, Matthew and Gee, Toby. "Moduli stacks of étale (φ, Γ)-modules and the existence of crystalline lifts." AMS-215, Annals of Math Studies 408.

[T1-3] Tong, Xin. "Analytic Geometry and Hodge-Frobenius Structure." arXiv:2011.08358.

[T2-3] Tong, Xin. "Geometric and Representation Theoretic Aspects of p-adic Motives." UCSD dissertation, 2021.

[KL1-3] Kedlaya, Kiran S. and Ruochuan Liu. "Finiteness of cohomology of local systems on rigid analytic spaces." arXiv:1611.06930.

[Kie-3] Kiehl, R. "Der Endlichkeitssatz für eigentliche Abbildungen in der nichtarchimedischen Funktionentheorie." Inventiones Mathematicae 2.3(1967):191-214.

[CKZ-3] Carter, A., K. S. Kedlaya and G. Zábrádi. "Drinfeld's lemma for perfectoid spaces and overconvergence of multivariate (φ, Γ)-modules." Documenta Mathematica, 26, 2021, 1329-1393.

[KL2-3] Kedlaya, K. S. and R. Liu. "Relative p-adic Hodge theory: Foundations." Astérisque-Société Mathematique de France 2015.371.

[KL3-3] Kedlaya, K. S. and R. Liu. "Relative p-adic Hodge theory II: Imperfect Period Rings." arXiv:1602.06899.

[KPX-3] Kedlaya, Kiran S., Jonathan Pottharst and Liang Xiao. "Cohomology of arithmetic families of (φ, Γ)-modules." Journal of the American Mathematical Society 27 (2014): 1043-1115.

[PZ-3] Pal, Aprameyo and Gergely Zábrádi. "Cohomology and Overconvergence for Representations of Powers of Galois Groups." Journal of the Institute of Mathematics of Jussieu 20 (2021): 361-421.

[KL4-3] Kedlaya, Kiran S. and Ruochuan Liu. "On families of (phi, Gamma)-modules." Algebra and Number Theory 4, 2010, page 943-967.

[FK-3] Fukaya, Takako and Kazuya Kato. "A formulation of conjectures on p-adic zeta functions in noncommutative Iwasawa theory." Proceedings of the St. Petersburg Mathematical Society, 2006, page 1-85.

[Ka1-3] Kato, Kazuya. "Lectures on the approach to Iwasawa theory for Hasse-Weil L-functions via B_{dR} . Part I." Arithmetic Algebraic geometry, Trento 1991, 50-163. Lecture Notes in Math 1553.

[Ka2-3] Kato, Kazuya. "Lectures on the approach to Iwasawa theory for Hasse-Weil L-functions via B_{dR} . Part II." unpblished manuscript.

[Na-3] Nakamura, Kentaro. "A generalization of Kato's local conjecture for (φ, Γ)-modules over the Robba ring." Algebra and Number Theory 11 (2017): 319-404.

[AB1-3] Andreatta, Fabrizio and Brinon, Olivier. (2010). "B_{dR}-représentations dans le cas relatif." Annales Scientifiques de l'École Normale Supérieure. 43. 10.24033/asens.2121.

[AB2-3] Brinon, Olivier and Fabrizio Andreatta. "Surconvergence des représentations p-adiques: le cas relatif." Astérisque 319 (2008): 39-116.

[AB3-3] Andreatta, Fabrizio and Brinon, Olivier. "Acyclicité geométrique de B_{cris}." Comment Math. Helv. 88, 2013, no 4, 965-1022.

[AI1-3] Andreatta, Fabrizio, and Iovita, Adrian. "Semistable Sheaves and Comparison Isomorphisms in the Semistable Case." Rendiconti del Seminario Matematico della Università di Padova 128 (2012): 131-286.

[AI2-3] Andreatta, F., and Iovita, A. (2013). "Comparison isomorphisms for smooth formal schemes." Journal of the Institute of Mathematics of Jussieu, 12(1), 77-151. doi:10.1017/S1474748012000643.

[AI3-3] Andreatta, Fabrizio and Adrian Iovita. "Global applications of relative (phi,Gamma)-modules I." Astérisque 319, 339-420, 2008.

[BBBK-3] Bambozzi, Federico, Oren Ben-Bassat and Kobi Kremnizer. "Analytic geometry over F_1 and the Fargues-Fontaine curve." Advances in Mathematics, vol 356 (2019).

[BBM-3] Ben-Bassat, Oren and Devarshi Mukherjee. "Analytification, localization and homotopy epimorphisms." Bulletin des Sciences Mathématiques, vol 176 (2022).

[BMS1-3] Bhatt, Bhargav, Matthew T. Morrow and Peter Scholze. "Integral p-adic Hodge theory." Publications mathématiques de l'IHÉS 128 (2018): 219-397.

[BMS2-3] Bhatt, Bhargav, Matthew T. Morrow and Peter Scholze. "Topological Hochschild homology and integral p-adic Hodge theory." Publications mathématiques de l'IHÉS 129 (2019): 199-310.

[BS1-3] Bhatt, Bhargav and Peter Scholze. "Prisms and Prismatic Cohomology." arXiv:1905.08229. To appear in Annals of Mathematics.

[BF1-3] Burns, David and Matthias Flach. "Tamagawa numbers for motives with (non-commutative) coefficients." Documenta Mathematica 6 (2001): 501-570.

[BF2-3] Burns, Dylan Michael and Matthias Flach. "Tamagawa numbers for motives with (noncommutative) coefficients, II." American Journal of Mathematics 125 (2003): 475-512.

[CS1-3] Clausen, Dustin and Peter Scholze. "Lectures on Condensed Mathematics." Http://www.math.uni-math.edu/people/scholze/Notes.html.

[CS2-3] Clausen, Dustin and Peter Scholze. "Lectures on Analytic Geometry." Http://www.math.uni-math.edu/people/scholze/Notes.html.

[Fa1-3] Faltings, Gerd. "p-adic Hodge theory." Journal of the American Mathematical Society 1 (1988): 255-299.

[Fa2-3] Faltings, Gerd. "Crystalline cohomology and p-adic Galois-representations." Algebraic Analysis, geometry and number theory, Baltimore MD 1988, p25-80, JHU press, Baltimore, MD, 1989.

[Fa3-3] Faltings, Gerd. "Almost étale extensions." Astérisque 279 (2002): 185-270.

[FF-3] Fargues, Laurent, Jean Marc Fontaine. "Courbes et fibrés vectoriels en théorie de Hodge p-adique." Astérisque 406 (2018): 1-382.

[FS-3] Fargues, Laurent and Peter Scholze. "Geomerization of Local Langlands Correspondence." arXiv:2102.13459.

[Fon1-3] Fontaine, Jean-Marc. "Le corps des périodes p-adiques, dans Périodes p-adiques." Séminaire de Bures, 1988, Astérisque, no. 223 (1994), Exposé no. 2, 43 p.

[Fon2-3] Fontaine, Jean-Marc. "Représentations p-adiques semi-stables." Astérisque 223, 1994, p113-184.

[Fon3-3] Fontaine, Jean-Marc. "Représentations ℓ-adiques potentiellement semi-stables." Astérisque 223, 1994, 321-347.

[Fon4-3] Fontaine, Jean-Marc. "Sur certains types de représentations p-adiques du groupe de Galois d'un corps local; construction d'un anneau de Barsotti-Tate." Annals of Mathematics 115 (1982): 529.

[Fon5-3] Fontaine, Jean Marc. "Cohomologie de De Rham, cohomologie cristalline et representations p-adiques." Algebraic geometry, Tokyo/Kyoto 1982, 1016, Lecture Notes in Mathematics, p86-108, 1983.

[Iwa-3] Iwasawa, K. "Analogies between number fields and function fields, some recent advances in basic sciences." Proc. Annual Sci. Conf. New York, 1965-1966, Vol 2, 203-208.

[KP-3] Kedlaya, Kiran S. and Jonathan Pottharst. "On categories of (phi,Gamma)-modules." Proceedings of Symposia in Pure Mathematics, Volume 97.2, 2018.

[KKM-3] Kelly, Jack, Kobi Kremnizer and Devarshi Mukherjee. "Analytic Hochschild-Kostant-Rosenberg Theorem." arxiv:2111.03502, 2021.

[Lu1-3] Lurie, Jacob. Higher Topos Theory (AM-170). Princeton: Princeton University Press, 2009. https://doi.org/10.1515/9781400830558.

[Lu2-3] Lurie, Jacob. "Higher Algebra." https://www.math.ias.edu/~lurie/.

[Lu3-3] Lurie, Jacob. "Spectral Algebraic Geometry." https://www.math.ias.edu/~lurie/.

[Mann-3] Mann, Lucas. "A p-adic 6-Functor Formalism in Rigid-Analytic Geometry." arXiv:2206.02022.

[Sch1-3] Scholze, P. (2013). "*p*-adic hodge theory for rigid-analytic varieties." Forum of Mathematics, Pi, 1, E1. doi:10.1017/fmp.2013.1.

[Sch2-3] Scholze, Peter. "Etale cohomology of diamonds." arXiv:1709.07343.

[Sch3-3] Scholze, Peter. "Perfectoid Spaces." Publications mathématiques de l'IHÉS 116 (2020): 245-313.

[BS2-3] Bhatt, Bhargav and Scholze, Peter. (2013). "The Pro-étale topology for schemes." Astérisque 369. 2015. 99-201.

[Wi1-3] Witte, Malte. "On a noncommutative Iwasawa main conjecture for varieties over finite fields." Journal of the European Mathematical Society 16.2 (2014): 289-325.

[Wi2-3] Witte, Malte. "Non-Commutative Iwasawa theory for global fields." Diss. Habilitation Thesis, University of Heidelberg, 2017.

[Wi3-3] Witte, Malte. "Noncommutative Iwasawa Main conjectures for varieties over finite fields." Dissertation.

[B-3] Bhatt, Bhargav. "p-adic derived de Rham cohomology." arXiv:1204.6560.

[Bei-3] Beilinson, Alexander. "p-adic periods and derived de Rham cohomology." Journal of the American Mathematical Society, vol 25, no. 3 2012, p715-738.

[Grot1-3] Grothendieck, A. "Crystals and the de Rham cohomology of schemes." Dix exposés sur la cohomologie des schemas. 306, 1968, 358.

[Grot2-3] Grothendieck, A. "Letter to Serre." 1964.

[SGAIV1-3] Bourbaki, N, M. Artin, A. Grothendieck, P. Deligne and Jean Louis Vertier. "Théorie des topos et cohomologie étale des schémas, tome I." SGA 1963-1964. Vol 1, Springer, 2006. Lecture Notes in Mathematics 269.

[SGAIV2-3] Artin, M, A. Grothendieck and Jean Louis Vertier. "Théorie des topos et cohomologie étale des schémas, tome II." SGA 1963-1964. Vol 270, Lecture Notes in Mathematics.

[SGAIV3-3] Deligne, P, M. Artin, B. Saint-Donat, A. Grothendieck and Jean Louis Vertier. "Théorie des topos et cohomologie étale des schémas, tome III." SGA 1963-1964. Vol 305, Lecture Notes in Mathematics.

[Guo-3] Guo, Haoyang. "Crystalline cohomology of rigid analytic spaces." (2021). arXiv:2112.14304.

[Ill1-3] Illusie, Luc. "Complexe Cotangent et Déformation I." 239, Lecture Notes in Mathematics. 1971.

[Ill2-3] Illusie, Luc. "Complexe Cotangent et Déformation II." 283, Lecture Notes in Mathematics. 1972.

[Ked1-3] Kedlaya, Kiran. "Sheaves, Stacks and Shtukas" Arizona Winter School lecture notes 2017, in Math Surveys and Monographs 242.

[CBCKSW-3] Cais, Bryden, Bhargav Bhatt, Ana Caraiani, Kiran Kedlaya, Peter Scholze and Jared Weinstein. "Perfectoid Spaces: Lectres from the 2017 Arizona Winter School." Math Surveys and Monogrphs 242, 2019.

[CS-3] H. Cartan and J.P. Serre. "Une théorèm de finitude concernant des variétés analytiques compactes." C. R. Acad. Sci. Paris, 237, 1953, page 128-130.

[T3-3] Tong, Xin. "Topologization and Functional Analytification I: Intrinsic Morphisms of Commutative Algebras." arXiv:2102.10766.

[T4-3] Tong, Xin. "Topologization and Functional Analytification II: ∞-Categorical Motivic Constructions for Homotopical Contexts." arXiv:2112.12679.

[EGH] Matthew Emerton, Toby Gee and Eugen Hellmann. "An introduction to categorical p-adic Langlands program." arXiv:2210.01404.

[D] Drinfeld, V. G. "Langlands' conjecture for GL_2 over functional fields." Proceedings of International Congress of Mathematicians, Helsinki, 1978. 565-574.

[L] Lafforgue, Laurent. 2002. "Chtoucas de Drinfeld et Correspondance de Langlands." Inventiones Mathematicae 147 (1): 1. https://doi.org/10.1007/s002220100174.

[HP] Hartl, Urs, and Richard Pink. 2004. "Vector Bundles with a Frobenius Structure on the Punctured Unit Disc." Compositio Mathematica 140 (3): 689-716. https://doi.org/10.1112/S0010437X03000216.

[HHS] Hellmann, Eugen, Valentin Hernandez and Benjamin Schraen. Work in progress.

[W] Evan B. Warner. "Adic Moduli Spaces." PhD Thesis, Stanford University. ProQuest, Ann Arbor, Michigan. 2017.